Performance Modelling of Computer Systems

Tribastone, Tschaikowski

Institut für Informatik
Ludwig-Maximilians-Universität München

Lumpability

Lumpability

Let X be a homogeneous, finite and irreducible CTMC with states $S = \{1, \ldots, n\}$ and let the transition rate from i to j be denoted by $q(i, j)$. Let $\chi = \{X_1, \ldots, X_N\}$ be a partition of the state space, with

$$X_i \subseteq S, \quad X_i \neq \emptyset, \text{ for all } i, \quad X_i \cap X_j = \emptyset, \text{ for all } i \neq j, \quad \text{and } \bigcup_{i=1}^{N} X_i = S.$$

We say that X is lumpable with respect to χ if, for any $X_i, X_j \in \chi$ and $i_1, i_2 \in X_i$, it holds that

$$\sum_{j \in X_j} q(i_1, j) = \sum_{j \in X_j} q(i_2, j).$$

Lumpability

Let X be a homogeneous, finite and irreducible CTMC with states $S = \{1, \ldots, n\}$ and let the transition rate from i to j be denoted by $q(i, j)$. Let $\chi = \{X_1, \ldots, X_N\}$ be a partition of the state space, with

$$X_i \subseteq S, \quad X_i \neq \emptyset, \text{ for all } i, \quad X_i \cap X_j = \emptyset, \text{ for all } i \neq j, \quad \text{and } \bigcup_{i=1}^{N} X_i = S.$$

We say that X is lumpable with respect to χ if, for any $X_i, X_j \in \chi$ and $i_1, i_2 \in X_i$, it holds that

$$\sum_{j \in X_j} q(i_1, j) = \sum_{j \in X_j} q(i_2, j).$$

Theorem

χ yields an aggregated (lumped) CTMC X' with states $\{X_1, \ldots, X_N\}$ and

$$q(X_i, X_j) := \sum_{j \in X_j} q(i, j), \quad \textit{for an arbitrary} \quad i \in X_i.$$

Lumpability

Theorem

Let $(\pi_1, \ldots, \pi_n)$ *denote the stationary distribution of* X *and* $(\pi'_1, \ldots, \pi'_N)$ *the stationary distribution of* X'. *Then, it holds that*

$$\pi'_j = \sum_{i \in X_j} \pi_i, \quad 1 \leq j \leq N.$$

Lumpability

Theorem

Let $(\pi_1, \ldots, \pi_n)$ denote the stationary distribution of X and $(\pi'_1, \ldots, \pi'_N)$ the stationary distribution of X'. Then, it holds that

$$\pi'_j = \sum_{i \in X_j} \pi_i, \quad 1 \leq j \leq N.$$

For instance, let us consider the model

$$C_0 \stackrel{def}{=} (r, u).C_1 \qquad C_1 \stackrel{def}{=} (t, v).C_0$$

$$S_0 \stackrel{def}{=} (r, w).S_1 \qquad S_1 \stackrel{def}{=} (i, x).S_0 \qquad Sys := C_0[N] \underset{\{r\}}{\bowtie} S_0[M] \; .$$

Lumpability

Theorem

Let $(\pi_1, \ldots, \pi_n)$ *denote the stationary distribution of* X *and* $(\pi'_1, \ldots, \pi'_N)$ *the stationary distribution of* X'. *Then, it holds that*

$$\pi'_j = \sum_{i \in X_j} \pi_i, \quad 1 \leq j \leq N.$$

For instance, let us consider the model

$$C_0 \stackrel{def}{=} (r, u).C_1 \qquad C_1 \stackrel{def}{=} (t, v).C_0$$

$$S_0 \stackrel{def}{=} (r, w).S_1 \qquad S_1 \stackrel{def}{=} (i, x).S_0 \qquad Sys := C_0[N] \underset{\{r\}}{\bowtie} S_0[M] \;.$$

- $|ds(Sys)| = 2^{N+M}$ grows exponentially in N and M. Numerical analysis via explicit state enumeration is infeasible.
- Often, for many performance analyses, it is sufficient to know the distributions of the populations of the sequential components C_0, C_1, S_0, and S_1.

Lumpability — An Example

Let us denote a state in $ds(Sys)$ by $(i_1, \ldots, i_N, j_1, \ldots, j_M) \in \{0,1\}^{N+M}$. Observing that the populations are given by the function

$$(i_1, \ldots, i_N, j_1, \ldots, j_M) \mapsto (N - \sum_{k=1}^{N} i_k, \sum_{k=1}^{N} i_k, M - \sum_{k=1}^{M} j_k, \sum_{k=1}^{M} j_k),$$

suggests to construct the partition from the equivalence relation $\sim \subseteq ds(Sys) \times ds(Sys)$ defined by

$$(i_1, \ldots, i_N, j_1, \ldots, j_M) \sim (i'_1, \ldots, i'_N, j'_1, \ldots, j'_M) :\Leftrightarrow \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} i'_k \ \wedge \ \sum_{k=1}^{M} j_k = \sum_{k=1}^{M} j'_k.$$

Lumpability — An Example

Let us denote a state in $ds(Sys)$ by $(i_1, \ldots, i_N, j_1, \ldots, j_M) \in \{0,1\}^{N+M}$. Observing that the populations are given by the function

$$(i_1, \ldots, i_N, j_1, \ldots, j_M) \mapsto (N - \sum_{k=1}^{N} i_k, \sum_{k=1}^{N} i_k, M - \sum_{k=1}^{M} j_k, \sum_{k=1}^{M} j_k),$$

suggests to construct the partition from the equivalence relation $\sim \subseteq ds(Sys) \times ds(Sys)$ defined by

$$(i_1, \ldots, i_N, j_1, \ldots, j_M) \sim (i'_1, \ldots, i'_N, j'_1, \ldots, j'_M) :\Leftrightarrow \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} i'_k \ \wedge \ \sum_{k=1}^{M} j_k = \sum_{k=1}^{M} j'_k.$$

Indeed, one can prove that the CTMC of *Sys* is lumpable with respect to the partition $ds(Sys)/\sim$.

Lumpability — An Example

Let us fix $X_i, X_j \in ds(Sys)/\sim$ and $i_1, i_2 \in X_i$. We have to show that

$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \overset{(!)}{=\!=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

Lumpability — An Example

Let us fix $X_i, X_j \in ds(Sys)/\sim$ and $i_1, i_2 \in X_i$. We have to show that

$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \overset{(!)}{=\!=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

Together with $k \in \{1,2\}, X_i = [(C_0^i, C_1^i, S_0^i, S_1^i)], X_j = [(C_0^j, C_1^j, S_0^j, S_1^j)]$ we can infer

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i - 1, C_1^i + 1, S_0^i - 1, S_1^i + 1)]$:

$$q(i_k, X_j) = \left(\frac{u}{uC_0^i} \cdot \frac{w}{wS_0^i} \min(uC_0^i, wS_0^i)\right) \cdot (C_0^i \cdot S_0^i) = \min(uC_0^i, wS_0^i)$$

Lumpability — An Example

Let us fix $X_i, X_j \in ds(Sys)/\sim$ and $i_1, i_2 \in X_i$. We have to show that

$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \overset{(!)}{=\!=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

Together with $k \in \{1, 2\}, X_i = [(C_0^i, C_1^i, S_0^i, S_1^i)], X_j = [(C_0^j, C_1^j, S_0^j, S_1^j)]$ we can infer

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i - 1, C_1^i + 1, S_0^i - 1, S_1^i + 1)]$:

$$q(i_k, X_j) = \left(\frac{u}{uC_0^i} \cdot \frac{w}{wS_0^i} \min(uC_0^i, wS_0^i) \right) \cdot (C_0^i \cdot S_0^i) = \min(uC_0^i, wS_0^i)$$

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i + 1, C_1^i - 1, S_0^i, S_1^i)]$: $q(i_k, X_j) = vC_1^i$

Lumpability — An Example

Let us fix $X_i, X_j \in ds(Sys)/\sim$ and $i_1, i_2 \in X_i$. We have to show that

$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \overset{(!)}{=\!=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

Together with $k \in \{1,2\}, X_i = [(C_0^i, C_1^i, S_0^i, S_1^i)], X_j = [(C_0^j, C_1^j, S_0^j, S_1^j)]$ we can infer

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i - 1, C_1^i + 1, S_0^i - 1, S_1^i + 1)]$:

$$q(i_k, X_j) = \left(\frac{u}{uC_0^i} \cdot \frac{w}{wS_0^i} \min(uC_0^i, wS_0^i) \right) \cdot (C_0^i \cdot S_0^i) = \min(uC_0^i, wS_0^i)$$

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i + 1, C_1^i - 1, S_0^i, S_1^i)]$: $q(i_k, X_j) = vC_1^i$
- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i, C_1^i, S_0^i + 1, S_1^i - 1)]$: $q(i_k, X_j) = xS_1^i$

Lumpability — An Example

Let us fix $X_i, X_j \in ds(Sys)/\sim$ and $i_1, i_2 \in X_i$. We have to show that

$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \overset{(!)}{=\!=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

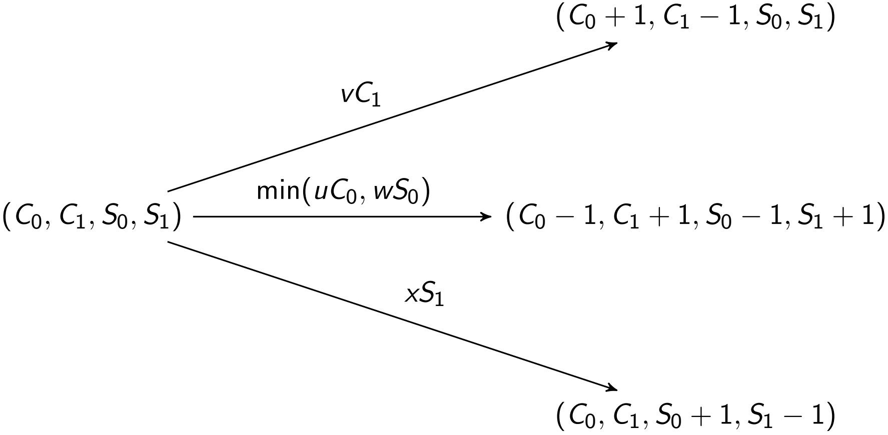
Together with $k \in \{1, 2\}, X_i = [(C_0^i, C_1^i, S_0^i, S_1^i)], X_j = [(C_0^j, C_1^j, S_0^j, S_1^j)]$ we can infer

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i - 1, C_1^i + 1, S_0^i - 1, S_1^i + 1)]$:

$$q(i_k, X_j) = \left(\frac{u}{uC_0^i} \cdot \frac{w}{wS_0^i} \min(uC_0^i, wS_0^i) \right) \cdot (C_0^i \cdot S_0^i) = \min(uC_0^i, wS_0^i)$$

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i + 1, C_1^i - 1, S_0^i, S_1^i)]$: $q(i_k, X_j) = vC_1^i$
- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i, C_1^i, S_0^i + 1, S_1^i - 1)]$: $q(i_k, X_j) = xS_1^i$
- otherwise: $q(i_k, X_j) = 0$

Lumpability — An Example



- The transitions out of a state in the lumped CTMC.
- The lumped CTMC has $(N+1)(M+1)$ states.

Strong Equivalence

Let us denote the actions of a PEPA process *Sys* by $\mathcal{A}(Sys)$. A relation $R \subseteq ds(Sys) \times ds(Sys)$ is a strong equivalence relation on $ds(Sys)$, if

$$\forall \alpha \in \mathcal{A}(Sys) \forall (P, Q) \in R \forall C \in ds(Sys)/R \quad \text{it holds}$$

$$\sum_{T \in C} q(P, T, \alpha) = \sum_{T \in C} q(Q, T, \alpha),$$

where $q(P_i, P_j, \alpha)$ denotes the α-transition rate from P_i to P_j in the CTMC of $ds(Sys)$.

Strong Equivalence — Facts

- The relation discussed in the lumpability example is a strong equivalence relation on $ds(C_0[N] \underset{\{r\}}{\bowtie} S_0[M])$.

Strong Equivalence — Facts

- The relation discussed in the lumpability example is a strong equivalence relation on $ds(C_0[N] \underset{\{r\}}{\bowtie} S_0[M])$.
- One can prove that the CTMC X of Sys is lumpable with respect to the partition $ds(Sys)/R$. Let us denote the corresponding lumped CTMC by X/R.

Strong Equivalence — Facts

- The relation discussed in the lumpability example is a strong equivalence relation on $ds(C_0[N] \underset{\{r\}}{\bowtie} S_0[M])$.
- One can prove that the CTMC X of Sys is lumpable with respect to the partition $ds(Sys)/R$. Let us denote the corresponding lumped CTMC by X/R.
- The above result leads to the following observation for two arbitrary PEPA processes Sys_1 and Sys_2: if we can find two strong equivalence relations R_1, R_2 on $ds(Sys_1)$, $ds(Sys_2)$, respectively, and there is a one-to-one correspondence between X_1/R_1 and X_2/R_2, then the stochastic behaviour of Sys_1 is given by that of Sys_2, and vice versa.

Strong Equivalence — An Example

For instance, let us consider the following model.

$$C_0 \stackrel{def}{=} (\alpha, 2r).C_1 \quad C_1 \stackrel{def}{=} (\beta, s).C_0$$

$$D_0 \stackrel{def}{=} (\alpha, 2r).D_1 \quad D_1 \stackrel{def}{=} (\beta, s).D_2 \quad D_2 \stackrel{def}{=} (\alpha, r).D_3 + (\alpha, r).D_1 \quad D_3 \stackrel{def}{=} (\beta, s).D_0$$

Strong Equivalence — An Example

For instance, let us consider the following model.

$$C_0 \stackrel{def}{=} (\alpha, 2r).C_1 \quad C_1 \stackrel{def}{=} (\beta, s).C_0$$

$$D_0 \stackrel{def}{=} (\alpha, 2r).D_1 \quad D_1 \stackrel{def}{=} (\beta, s).D_2 \quad D_2 \stackrel{def}{=} (\alpha, r).D_3 + (\alpha, r).D_1 \quad D_3 \stackrel{def}{=} (\beta, s).D_0$$

It is clear that $R_C = \{\{C_0\}, \{C_1\}\}$ is a strong equivalence relation on $ds(C_0)$ and $R_D = \{\{D_0, D_2\}, \{D_1, D_3\}\}$ a strong equivalence relation of $ds(D_0)$.

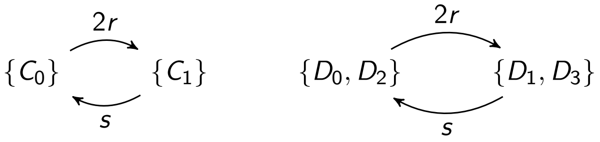
Strong Equivalence — An Example

For instance, let us consider the following model.

$$C_0 \stackrel{def}{=} (\alpha, 2r).C_1 \quad C_1 \stackrel{def}{=} (\beta, s).C_0$$

$$D_0 \stackrel{def}{=} (\alpha, 2r).D_1 \quad D_1 \stackrel{def}{=} (\beta, s).D_2 \quad D_2 \stackrel{def}{=} (\alpha, r).D_3 + (\alpha, r).D_1 \quad D_3 \stackrel{def}{=} (\beta, s).D_0$$

It is clear that $R_C = \{\{C_0\}, \{C_1\}\}$ is a strong equivalence relation on $ds(C_0)$ and $R_D = \{\{D_0, D_2\}, \{D_1, D_3\}\}$ a strong equivalence relation of $ds(D_0)$. Since the two lumped CTMCs



are in a one-to-one correspondence (i.e., they are equal up to a relabelling of the nodes), we conclude that the stochastic behaviour of C_0 is given by that of D_0, and vice versa.