Performance Modelling of Computer Systems

Mirco Tribastone

Institut für Informatik
Ludwig-Maximilians-Universität München

Stochastic Process Algebra

Overview

- Overview of classic (untimed) process algebra
- Associating exponential distributions to activities
- Introduction to the stochastic process algebra PEPA

Bibliographic references:

- J. Hillston. **A Compositional Approach to Performance Modelling**. Cambridge University Press, 1996.
- A. Clark, J. Hillston, and M. Tribastone. **Stochastic Process Algebras**. In *Formal Methods for Performance Evaluation: the 7th International School on Formal Methods for the Design of Computer, Communication, and Software Systems, SFM 2007*, LNCS 4486, Springer-Verlag.

Features of Stochastic Process Algebra

- A high-level description technique for continuous-time Markov chains...
- ...but not only:
  - hybrid systems;
  - continuous-state systems;
  - ...
- A formal method: a textual language with a precise syntax and semantics.
- A compositional approach to performance evaluation: the modelling and reasoning is modular.

- Models are constructed from components which engage in activities.

$$(\alpha, r).P$$

action type or name

activity rate (parameter of an exponential distribution)

component/ derivative

- The language is used to generate a CTMC for performance modelling.

PEPA

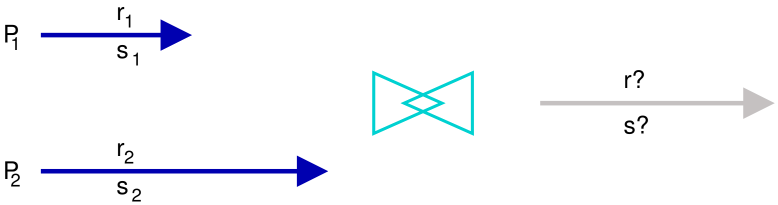
BNF Syntax

$$S ::= (\alpha, r).S \mid S + S \mid A$$
$$P ::= S \mid P \underset{L}{\bowtie} P \mid P/L$$

PREFIX:	$(\alpha, r).S$	designated first action
CHOICE:	$S + S$	competing components (race policy)
CONSTANT:	$A \stackrel{def}{=} S$	assigning names
COOPERATION:	$P \underset{L}{\bowtie} P$	$\alpha \notin L$ concurrent activity (*individual actions*) $\alpha \in L$ cooperative activity (*shared actions*)
HIDING:	P/L	abstraction $\alpha \in L \Rightarrow \alpha \rightarrow \tau$

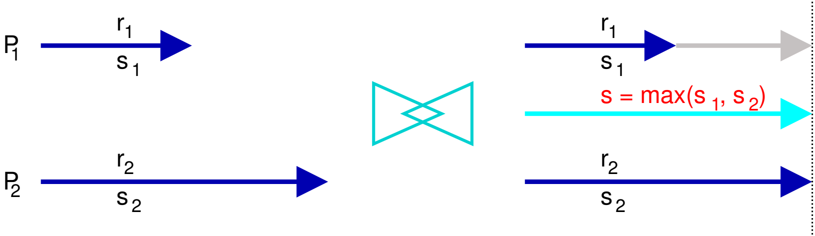
Timed Synchronisation

- The issue of what it means for two timed activities to synchronise is a vexed one...



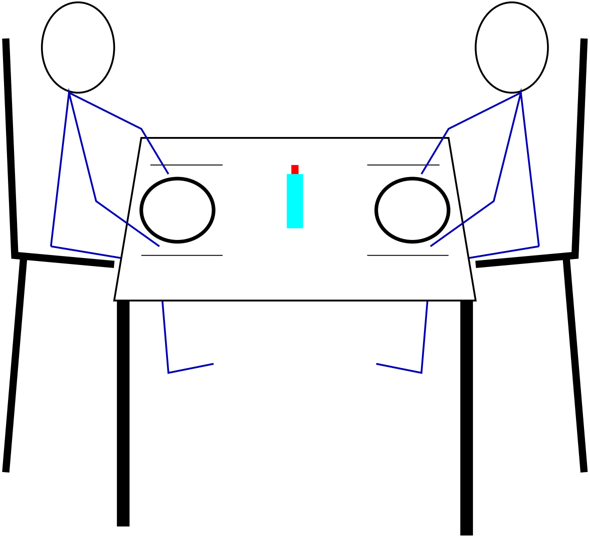
Timed Synchronisation

- The issue of what it means for two timed activities to synchronise is a vexed one. . .



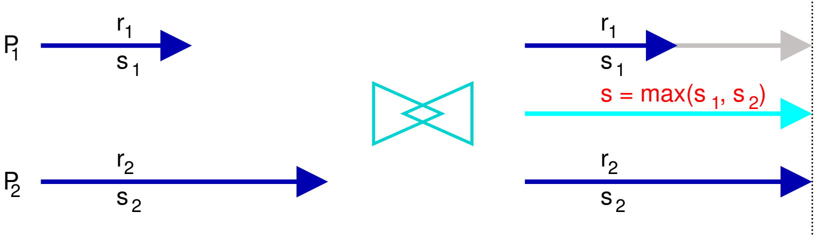
Barrier Synchronisation

Timed Synchronisation



Timed Synchronisation

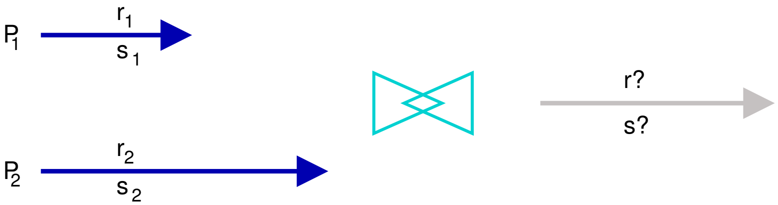
- The issue of what it means for two timed activities to synchronise is a vexed one. . .



s is no longer exponentially distributed

Timed Synchronisation

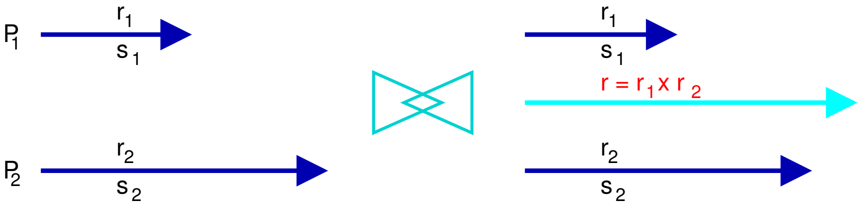
- The issue of what it means for two timed activities to synchronise is a vexed one. . .



algebraic considerations limit choices

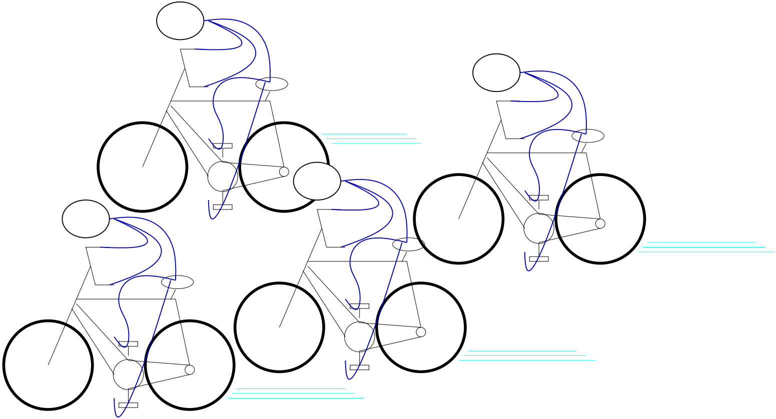
Timed Synchronisation

- The issue of what it means for two timed activities to synchronise is a vexed one. . .



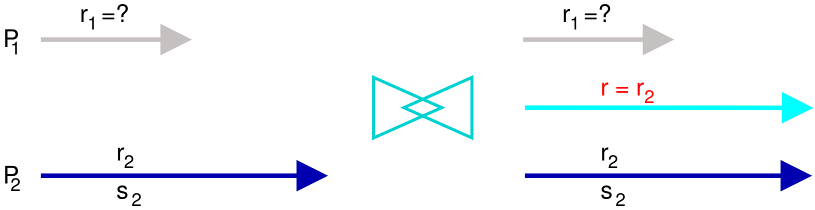
TIPP: new rate is product of individual rates

Timed Synchronisation



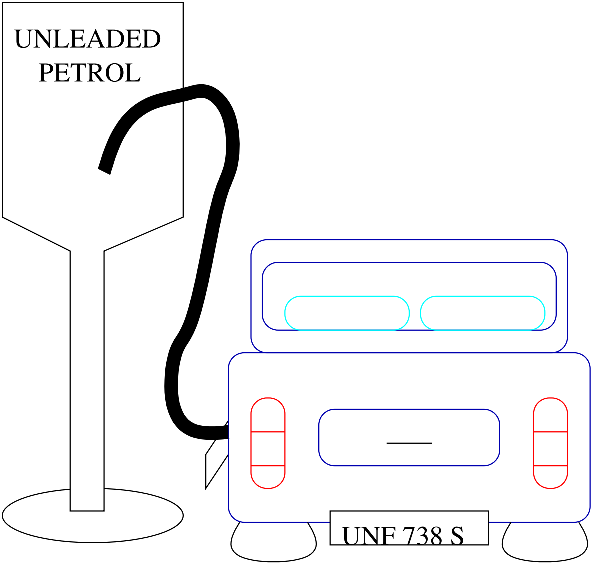
Timed Synchronisation

- The issue of what it means for two timed activities to synchronise is a vexed one. . .



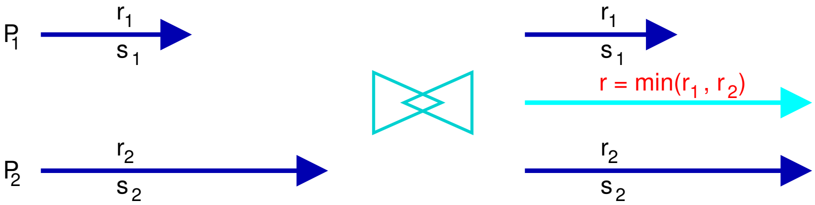
EMPA: one participant is passive

Timed Synchronisation



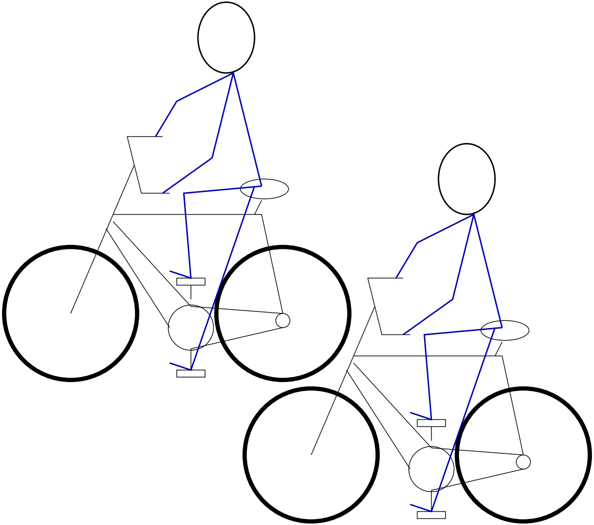
Timed Synchronisation

- The issue of what it means for two timed activities to synchronise is a vexed one. . .



bounded capacity: new rate is the minimum of the rates

Timed Synchronisation



Cooperation in PEPA

- In PEPA each component has a bounded capacity to carry out activities of any particular type, determined by the apparent rate for that type.
- Synchronisation, or cooperation cannot make a component exceed its bounded capacity.
- Thus the apparent rate of a cooperation is the minimum of the apparent rates of the co-operands.

Operational Semantics of PEPA

$$\mathrm{S}_0: \quad \frac{}{(\alpha,r).P \xrightarrow{(\alpha,r)} P}$$

$$\mathrm{A}_0: \quad \frac{P \xrightarrow{(\alpha,r)} P'}{A \xrightarrow{(\alpha,r)} P'}, \ A \stackrel{def}{=} P$$

$$\mathrm{S}_1: \quad \frac{P \xrightarrow{(\alpha,r)} P'}{P+Q \xrightarrow{(\alpha,r)} P'}$$

$$\mathrm{S}_2: \quad \frac{Q \xrightarrow{(\alpha,r)} Q'}{P+Q \xrightarrow{(\alpha,r)} Q'}$$

$$\mathrm{C}_0: \quad \frac{P \xrightarrow{(\alpha,r)} P'}{P \underset{L}{\bowtie} Q \xrightarrow{(\alpha,r)} P' \underset{L}{\bowtie} Q}, \ \alpha \notin L$$

$$\mathrm{C}_1: \quad \frac{Q \xrightarrow{(\alpha,r)} Q'}{P \underset{L}{\bowtie} Q \xrightarrow{(\alpha,r)} P \underset{L}{\bowtie} Q'}, \ \alpha \notin L$$

$$\mathrm{C}_2: \quad \frac{P \xrightarrow{(\alpha,r_1)} P' \qquad Q \xrightarrow{(\alpha,r_2)} Q'}{P \underset{L}{\bowtie} Q \xrightarrow{(\alpha,R)} P' \underset{L}{\bowtie} Q'}, \ \alpha \in L$$

$$R = \frac{r_1}{r_\alpha(P)} \frac{r_2}{r_\alpha(Q)} \min\left(r_\alpha(P), r_\alpha(Q)\right)$$

$$\mathrm{H}_0: \quad \frac{P \xrightarrow{(\alpha,r)} P'}{P/L \xrightarrow{(\alpha,r)} P'/L}, \ \alpha \notin L$$

$$\mathrm{H}_1: \quad \frac{P \xrightarrow{(\alpha,r)} P'}{P/L \xrightarrow{(\tau,r)} P'/L}, \ \alpha \in L$$

Multiway Synchronisation

$$F \overset{def}{=} (fork, r_f).(join, r_j).F'$$

$$W_1 \overset{def}{=} (fork, r_{f_1}).(doWork_1, r_1).W_1'$$

$$W_2 \overset{def}{=} (fork, r_{f_2}).(doWork_2, r_2).W_2'$$

$$F' \overset{def}{=} \ldots, \quad W_1' \overset{def}{=} \ldots, \quad W_2' \overset{def}{=} \ldots$$

$$System \overset{def}{=} \left(F \underset{\{fork\}}{\bowtie} W_1\right) \underset{\{fork\}}{\bowtie} W_2$$

$$\frac{P \xrightarrow{(\alpha, r)} P'}{A \xrightarrow{(\alpha, r)} P'}, \; A \overset{def}{=} P \Longrightarrow$$

1 $$\frac{(fork, r_f).(join, r_j).F' \xrightarrow{(fork, r_f)} (join, r_j).F'}{F \xrightarrow{(fork, r_f)} (join, r_j).F'}$$

2 $$\frac{(fork, r_{f_1}).(doWork_1, r_1).W_1' \xrightarrow{(fork, r_{f_1})} (doWork_1, r_1).W_1'}{W_1 \xrightarrow{(fork, r_{f_1})} (doWork_1, r_1).W_1'}$$

3 $$\frac{(fork, r_{f_2}).(doWork_2, r_2).W_2' \xrightarrow{(fork, r_{f_2})} (doWork_2, r_2).W_2'}{W_2 \xrightarrow{(fork, r_{f_2})} (doWork_2, r_2).W_2'}$$

Multiway Synchronisation

$$F \stackrel{def}{=} (fork, r_f).(join, r_j).F'$$

$$W_1 \stackrel{def}{=} (fork, r_{f_1}).(doWork_1, r_1).W_1'$$

$$W_2 \stackrel{def}{=} (fork, r_{f_2}).(doWork_2, r_2).W_2'$$

$$F' \stackrel{def}{=} \ldots, \quad W_1' \stackrel{def}{=} \ldots, \quad W_2' \stackrel{def}{=} \ldots$$

$$System \stackrel{def}{=} \left(F \underset{\{fork\}}{\bowtie} W_1\right) \underset{\{fork\}}{\bowtie} W_2$$

$$\frac{F \xrightarrow{(fork,r_f)} (join, r_j)F' \qquad W_1 \xrightarrow{(fork,r_{f_1})} (doWork_1, r_1).W_1'}{F \underset{\{fork\}}{\bowtie} W_1 \xrightarrow{(fork,r')} (join, r_j).F' \underset{\{fork\}}{\bowtie} (doWork_1, r_1).W_1' \equiv LHS}$$

$$\frac{LHS \qquad W_2 \xrightarrow{(fork,r_{f_2})} (doWork_2, r_2).W_2'}{F \underset{\{fork\}}{\bowtie} W_1 \underset{\{fork\}}{\bowtie} W_2 \xrightarrow{(fork,r'')} (join, r_j).F' \underset{\{fork\}}{\bowtie} (doWork_1, r_1)W_1' \underset{\{fork\}}{\bowtie} (doWork_2, r_2).W_2'}$$

Other Communication Patterns

$$
\begin{aligned}
Premium &\stackrel{def}{=} (dwn, r_p).Premium' \\
Basic &\stackrel{def}{=} (dwn, r_b).Basic' \\
S &\stackrel{def}{=} (dwn, r_s).S' \\
&\dots \\
System &\stackrel{def}{=} (Premium \parallel Basic) \bowtie_L S, \\
L &= \{dwn\}
\end{aligned}
$$

$$\frac{P \xrightarrow{(\alpha,r)} P'}{P \bowtie_L Q \xrightarrow{(\alpha,r)} P' \bowtie_L Q}, \alpha \notin L$$

$$\frac{Q \xrightarrow{(\alpha,r)} Q'}{P \bowtie_L Q \xrightarrow{(\alpha,r)} P \bowtie_L Q'}, \alpha \notin L$$

$$\frac{P \xrightarrow{(\alpha,r_1)} P' \quad Q \xrightarrow{(\alpha,r_2)} Q'}{P \bowtie_L Q \xrightarrow{(\alpha,R)} P' \bowtie_L Q'}, \alpha \in L$$

$$R = \frac{r_1}{r_\alpha(P)} \frac{r_2}{r_\alpha(Q)} \min\left(r_\alpha(P), r_\alpha(Q)\right)$$

$$\cfrac{\cfrac{\cfrac{Premium \xrightarrow{(dwn,r_p)} Premium'}{Premium \parallel Basic \xrightarrow{(dwn,r_p)} Premium' \parallel Basic} \qquad \cfrac{}{S \xrightarrow{(dwn,r_s)} S'}}{Premium \parallel Basic \bowtie_L S \xrightarrow{(dwn,{\color{red}r_{ps}})} Premium' \parallel Basic \bowtie_L S'}}{{\color{red}System \xrightarrow{(dwn,r_{ps})} Premium' \parallel Basic \bowtie_L S'}}$$

(dwn, r_b)

Rates in PEPA

PEPA supports the notion of infinite capacity:

$$(\alpha, r).P, \qquad \text{with } r \in \mathbb{R}_{>0} \cup \{n\top, n \in \mathbb{N}\}.$$

- A positive real denotes the rate of the exponential distribution associated with the activity.
- The top symbol $\top$ denotes an unspecified (or passive) rate. The rate will be assigned by other cooperating components in the system.
- Passive rates are given weights (naturals) which are useful to determine the relative probabilities of distinct passive activities to occur. ($1\top$ is usually written $\top$ for short.)

Arithmetic for Passive Rates

$$
\begin{aligned}
m\top + n\top &= (m+n)\top, && \text{for any } m, n \in \mathbb{N} \\
\frac{m\top}{n\top} &= \frac{m}{n}, && \text{for any } m, n \in \mathbb{N} \\
\min(r, n\top) &= r, && \text{for any } r \in \mathbb{R}_{>0} \text{ and } n \in \mathbb{N} \\
\min(m\top, n\top) &= \min(m, n)\top, && \text{for any } m, n \in \mathbb{N}
\end{aligned}
$$

- Summation and division between active and passive rates are not allowed.
- For expression of the following kind:

$$\frac{r}{s} \times \frac{m\top}{n\top}, \qquad r, s \in \mathbb{R}_{>0}, m, n \in \mathbb{N}$$

we assume that the two divisions have precedence over the multiplication.

Apparent Rate Calculation

$$\frac{P \xrightarrow{(\alpha,r_1)} P' \quad Q \xrightarrow{(\alpha,r_2)} Q'}{P \underset{L}{\bowtie} Q \xrightarrow{(\alpha,R)} P' \underset{L}{\bowtie} Q'}, \alpha \in L, \qquad R = \frac{r_1}{r_\alpha(P)} \frac{r_2}{r_\alpha(Q)} \min\left(r_\alpha(P), r_\alpha(Q)\right)$$

$$r_\alpha\big(\,(\beta, r)\,.P\big) = \begin{cases} r & \text{if} \quad \beta = \alpha \\ 0 & \text{if} \quad \beta \neq \alpha \end{cases}$$

$$r_\alpha(P + Q) = r_\alpha(P) + r_\alpha(Q)$$

$$r_\alpha(P \underset{L}{\bowtie} Q) = \begin{cases} \min\big(r_\alpha(P), r_\alpha(Q)\big) & \text{if } \alpha \in L \\ r_\alpha(P) + r_\alpha(Q) & \text{if } \alpha \notin L \end{cases}$$

$$r_\alpha(P/L) = \begin{cases} r_\alpha(P) & \text{if} \quad \alpha \notin L \\ 0 & \text{if} \quad \alpha \in L \end{cases}$$

Components which are both active and passive with respect to some action type are not allowed, e.g. $(\alpha, 1.0).P + (\alpha, \top).P$.

Examples

For r_1, r_2 positive reals,

$$\frac{(\alpha, r_1).P_1 \xrightarrow{(\alpha,r_1)} P_1 \qquad (\alpha, r_2).P_2 \xrightarrow{(\alpha,r_2)} P_2}{(\alpha, r_1).P_1 \underset{\{\alpha\}}{\bowtie} (\alpha, r_2).P_2 \xrightarrow{(\alpha,R)} P_1 \underset{\{\alpha\}}{\bowtie} P_2},$$

where

$$\begin{aligned} R &= \frac{r_1}{r_\alpha\big((\alpha, r_1).P_1\big)} \frac{r_2}{r_\alpha\big((\alpha, r_2).P_2\big)} \min\Big(r_\alpha\big((\alpha, r_1).P_1\big), r_\alpha\big((\alpha, r_2).P_2\big)\Big) \\ &= \frac{r_1}{r_1}\frac{r_2}{r_2}\min(r_1, r_2) = \min(r_1, r_2). \end{aligned}$$

We recover the intuitive definition of the minimum between the two rates.

Examples

For r a positive real,

$$\frac{(\alpha, r).P_1 \xrightarrow{(\alpha,r)} P_1 \qquad (\alpha, \top).P_2 \xrightarrow{(\alpha,\top)} P_2}{(\alpha, r).P_1 \underset{\{\alpha\}}{\bowtie} (\alpha, \top).P_2 \xrightarrow{(\alpha,R)} P_1 \underset{\{\alpha\}}{\bowtie} P_2},$$

where

$$\begin{aligned} R &= \frac{r}{r_\alpha\big((\alpha, r).P_1\big)} \frac{\top}{r_\alpha\big((\alpha, \top).P_2\big)} \min\Big(r_\alpha\big((\alpha, r).P_1\big), r_\alpha\big((\alpha, \top).P_2\big)\Big) \\ &= \frac{r}{r}\frac{\top}{\top}\min(r, \top) = r. \end{aligned}$$

We recover the intuitive definition of infinite capacity — the rate of synchronisation is determined by the active component.

Examples

For r a positive real and any natural n,

$$\frac{(\alpha, r).P_1 \xrightarrow{(\alpha,r)} P_1 \qquad (\alpha, n\top).P_2 \xrightarrow{(\alpha,n\top)} P_2}{(\alpha, r).P_1 \underset{\{\alpha\}}{\bowtie} (\alpha, n\top).P_2 \xrightarrow{(\alpha,R)} P_1 \underset{\{\alpha\}}{\bowtie} P_2},$$

where

$$\begin{aligned} R &= \frac{r}{r_\alpha\big((\alpha, r).P_1\big)} \frac{n\top}{r_\alpha\big((\alpha, n\top).P_2\big)} \min\Big(r_\alpha\big((\alpha, r).P_1\big), r_\alpha\big((\alpha, n\top).P_2\big)\Big) \\ &= \frac{r}{r} \frac{n\top}{n\top} \min(r, n\top) = r. \end{aligned}$$

Passive weights may not affect the overall rate if only one passive component is present.

(Slightly More Complicated) Examples

$$
\boxed{
\begin{aligned}
Act &\stackrel{def}{=} (\alpha, r).Act' \\
Pas &\stackrel{def}{=} (\alpha, 1\top).Pas' + (\alpha, 2\top).Pas'' \\
Sys &\stackrel{def}{=} Act \underset{\{\alpha\}}{\bowtie} Pas
\end{aligned}
}
$$

$$
\frac{
\dfrac{
\dfrac{}{(\alpha, r).Act' \xrightarrow{(\alpha,r)} Act'}
}{
Act \xrightarrow{(\alpha,r)} Act'
}
\qquad
\dfrac{
\dfrac{
\dfrac{}{(\alpha, 1\top).Pas' \xrightarrow{(\alpha,1\top)} Pas'}
}{
(\alpha, 1\top).Pas' + (\alpha, 2\top).Pas'' \xrightarrow{(\alpha,1\top)} Pas'
}
}{
Pas \xrightarrow{(\alpha,1\top)} Pas'
}
}{
\dfrac{
Act \underset{\{\alpha\}}{\bowtie} Pas \xrightarrow{(\alpha,R')} Act' \underset{\{\alpha\}}{\bowtie} Pas'
}{
Sys \xrightarrow{(\alpha,R')} Act' \underset{\{\alpha\}}{\bowtie} Pas'
}
},
$$

$$
R' = \frac{r}{r_\alpha(Act)} \frac{1\top}{r_\alpha(Pas)} \min\left(r_\alpha(Act), r_\alpha(Pas)\right) = \frac{r}{r} \frac{1\top}{1\top + 2\top} \min(r, 1\top + 2\top) = \frac{1}{3} r.
$$

(Slightly More Complicated) Examples

$$Act \overset{def}{=} (\alpha, r).Act'$$
$$Pas \overset{def}{=} (\alpha, 1\top).Pas' + (\alpha, 2\top).Pas''$$
$$Sys \overset{def}{=} Act \underset{\{\alpha\}}{\bowtie} Pas$$

It is also possible to prove the following derivation tree:

$$\dfrac{\dfrac{\dfrac{}{(\alpha, r).Act' \xrightarrow{(\alpha,r)} Act'}}{Act \xrightarrow{(\alpha,r)} Act'} \qquad \dfrac{\dfrac{\dfrac{}{(\alpha, 2\top).Pas'' \xrightarrow{(\alpha,2\top)} Pas''}}{(\alpha, 1\top).Pas' + (\alpha, 2\top).Pas'' \xrightarrow{(\alpha,2\top)} Pas''}}{Pas \xrightarrow{(\alpha,2\top)} Pas''}}{\dfrac{Act \underset{\{\alpha\}}{\bowtie} Pas \xrightarrow{(\alpha,R'')} Act' \underset{\{\alpha\}}{\bowtie} Pas''}{Sys \xrightarrow{(\alpha,R'')} Act' \underset{\{\alpha\}}{\bowtie} Pas''}},$$

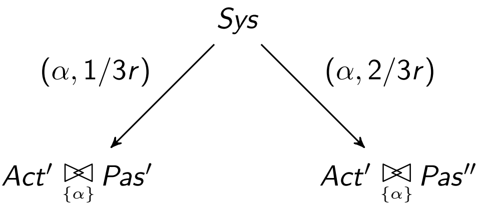
$$R'' = \frac{r}{r_\alpha(Act)} \frac{2\top}{r_\alpha(Pas)} \min\left(r_\alpha(Act), r_\alpha(Pas)\right) = \frac{r}{r} \frac{2\top}{1\top + 2\top} \min(r, 1\top + 2\top) = \frac{2}{3} r.$$

(Slightly More Complicated) Examples

$$Act \stackrel{def}{=} (\alpha, r).Act'$$

$$Pas \stackrel{def}{=} (\alpha, 1\top).Pas' + (\alpha, 2\top).Pas''$$

$$Sys \stackrel{def}{=} Act \underset{\{\alpha\}}{\bowtie} Pas$$



Apparent Rates in Active Cooperation

$$Cli \stackrel{def}{=} (\alpha, r_d).Cli'$$

$$Ser \stackrel{def}{=} (\alpha, r_u).Ser'$$

$$Sys \stackrel{def}{=} (Cli \parallel Cli) \underset{\{\alpha\}}{\bowtie} Ser$$

$$\frac{\dfrac{\dfrac{}{(\alpha, r_d).Cli' \xrightarrow{(\alpha, r_d)} Cli'}}{Cli \xrightarrow{(\alpha, r_d)} Cli'}}{Cli \parallel Cli \xrightarrow{(\alpha, r_d)} Cli' \parallel Cli} \qquad \frac{(\alpha, r_u).Ser' \xrightarrow{(\alpha, r_u)} Ser'}{Ser \xrightarrow{(\alpha, r_u)} Ser'}$$

$$\overline{Cli \parallel Cli \underset{\{\alpha\}}{\bowtie} Ser \xrightarrow{(\alpha, R')} Cli' \parallel Cli \underset{\{\alpha\}}{\bowtie} Ser'},$$

$$R' = \frac{r_d}{r_d + r_d} \frac{r_u}{r_u} \min(r_d + r_d, r_u) = \frac{1}{2} \min(r_d + r_d, r_u)$$

Apparent Rates in Active Cooperation

$$Cli \stackrel{def}{=} (\alpha, r_d).Cli'$$
$$Ser \stackrel{def}{=} (\alpha, r_u).Ser'$$
$$Sys \stackrel{def}{=} (Cli \parallel Cli) \underset{\{\alpha\}}{\bowtie} Ser$$

The following derivation tree can also be proven:

$$\dfrac{\dfrac{\dfrac{\overline{(\alpha, r_d).Cli' \xrightarrow{(\alpha, r_d)} Cli'}}{Cli \xrightarrow{(\alpha, r_d)} Cli'}}{\color{red}{Cli \parallel Cli \xrightarrow{(\alpha, r_d)} Cli \parallel Cli'}} \qquad \dfrac{(\alpha, r_u).Ser' \xrightarrow{(\alpha, r_u)} Ser'}{Ser \xrightarrow{(\alpha, r_u)} Ser'}}{Cli \parallel Cli \underset{\{\alpha\}}{\bowtie} Ser \xrightarrow{(\alpha, R'')} Cli \parallel Cli' \underset{\{\alpha\}}{\bowtie} Ser'},$$

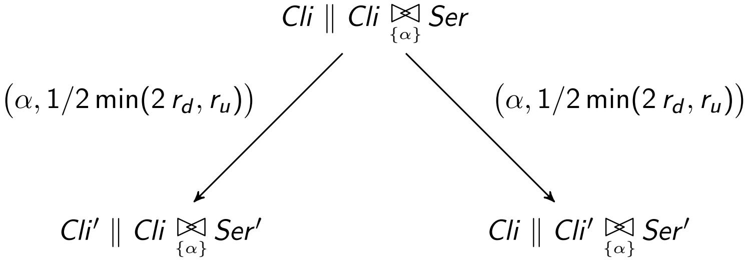
$$R'' = \frac{r_d}{r_d + r_d} \frac{r_u}{r_u} \min(r_d + r_d, r_u) = \frac{1}{2} \min(r_d + r_d, r_u) = R'$$

Apparent Rates in Active Cooperation

$$Cli \stackrel{def}{=} (\alpha, r_d).Cli'$$

$$Ser \stackrel{def}{=} (\alpha, r_u).Ser'$$

$$Sys \stackrel{def}{=} (Cli \parallel Cli) \underset{\{\alpha\}}{\bowtie} Ser$$



Labelled Transition System: Details

Derivative Set

Given a PEPA component P, the derivative set of P, denoted by $ds(P)$ is defined as the smallest set of components such that

- $P \in ds(P)$;
- if $P \xrightarrow{(\alpha,r)} P'$ then $P' \in ds(P)$.

Derivation Graph

Let $\mathcal{A}$ be a set of action labels and $\mathcal{A}ct = \{\mid (\alpha, r) : \alpha \in \mathcal{A}, r \in \mathbb{R}_{>0} \mid\}$. The derivation graph of a component P has $ds(P)$ as the set of nodes.
The multiset of arcs $A \in ds(P) \times ds(P) \times \mathcal{A}ct$ is such that

$$P \xrightarrow{(\alpha,r)} P' \implies \big(P, P', (\alpha, r)\big) \in A,$$

with multiplicity equal to the number of distinct derivations $P \xrightarrow{(\alpha,r)} P'$.

Why Multisets

$$P \stackrel{def}{=} (\alpha, r).P' \;\Big|\; P \stackrel{def}{=} (\alpha, r).P' + (\alpha, r).P' \;\Big|\; \ldots \;\Big|\; P \stackrel{def}{=} \sum_n (\alpha, r).P'$$

- If distinct inference trees were not taken into account, then the derivation graph would have only one transition $P \xrightarrow{(\alpha,r)} P'$.
- With a multiset, we have one, two, . . . , n such transitions, respectively.
- Intuitively, this capture the fact that process P has different apparent rates in these cases.

An Algorithm for State-Space Derivation

```
ds(P_0) ⇐ {P_0}
push P_0 onto Stack
while Stack is not empty do
   pop P off Stack
   infer multiset (P, P', (α, r)) from P
   for all (P, P', (α, r)) do
      if P' ∉ ds(P_0) then
         push P' onto Stack
         add P' to ds(P_0)
      end if
   end for
end while
```

The Underlying Markov Process

- Let P_0 be the initial state of the system.
- Assign a state to each process in $\mathrm{ds}(P_0)$.
- For each triple $\big(P, P', (\alpha, r)\big)$ with multiplicity m, assign rate $m\,r$ to the transition between P and P'.

Well-Formedness

- Note that all leaves of the derivation trees must have rates in the (strictly) positive reals.
- This means that passive actions must eventually synchronise with an active ones.
- Models that do not satisfy this condition are rejected.
- For example,

$$(\alpha, \top).P \underset{\{\alpha\}}{\bowtie} (\alpha, \top).Q$$

will be rejected for any P and Q.

Consumer/Producer in PEPA

$$
\begin{aligned}
Cons_1 &\overset{def}{=} (get, r_g).Cons_2 \\
Cons_2 &\overset{def}{=} (cons, r_c).Cons_1 \\
Prod_1 &\overset{def}{=} (make, r_m).Prod_2 \\
Prod_2 &\overset{def}{=} (put, r_p).Prod_1 \\
Buf_2 &\overset{def}{=} (get, \top).Buf_1 \\
Buf_1 &\overset{def}{=} (get, \top).Buf_0 \\
&\quad + (put, \top).Buf_2 \\
Buf_0 &\overset{def}{=} (put, \top).Buf_1 \\
Sys &\overset{def}{=} Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1
\end{aligned}
$$

Possible variants:

- A buffer with n places:

$$
\begin{aligned}
Buf_n &\overset{def}{=} (get, \top).Buf_{n-1} \\
Buf_i &\overset{def}{=} (get, \top).Buf_{i-1} \\
&\quad + (put, \top).Buf_{i+1}, \\
&\quad \text{for } 1 \leq i \leq n-1 \\
Buf_0 &\overset{def}{=} (put, \top).Buf_1
\end{aligned}
$$

- and k consumers:

$$
\overbrace{Cons_1 \parallel Cons_1 \parallel \ldots \parallel Cons_1}^{k} \underset{\{get\}}{\bowtie} Buf_n \underset{\{put\}}{\bowtie} Prod_1
$$

Consumer/Producer in PEPA

$$
\begin{array}{llllll}
Cons_1 & \stackrel{def}{=} & (get, r_g).Cons_2 & Prod_1 & \stackrel{def}{=} & (make, r_m).Prod_2 \\
Cons_2 & \stackrel{def}{=} & (cons, r_c).Cons_1 & Prod_2 & \stackrel{def}{=} & (put, r_p).Prod_1 \\
Buf_2 & \stackrel{def}{=} & (get, \top).Buf_1 & Buf_1 & \stackrel{def}{=} & (get, \top).Buf_0 + (put, \top).Buf_2 \\
Buf_0 & \stackrel{def}{=} & (put, \top).Buf_1 & Sys & \stackrel{def}{=} & Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1
\end{array}
$$

$$
\cfrac{\cfrac{\cfrac{Cons_1 \xrightarrow{(get, r_g)} Cons_2 \qquad Buf_2 \xrightarrow{(get, \top)} Buf_1}{Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \xrightarrow{(get, r_g)} Cons_2 \underset{\{get\}}{\bowtie} Buf_1}}{Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1 \xrightarrow{(get, r_g)} Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1}}{Sys \xrightarrow{(get, r_g)} Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1}
$$

Can we prove anything else for *Sys*?

Consumer/Producer in PEPA

$$\begin{array}{lcl} Cons_1 & \stackrel{def}{=} & (get, r_g).Cons_2 \\ Cons_2 & \stackrel{def}{=} & (cons, r_c).Cons_1 \\ Buf_2 & \stackrel{def}{=} & (get, \top).Buf_1 \\ Buf_0 & \stackrel{def}{=} & (put, \top).Buf_1 \end{array} \qquad \begin{array}{lcl} Prod_1 & \stackrel{def}{=} & (make, r_m).Prod_2 \\ Prod_2 & \stackrel{def}{=} & (put, r_p).Prod_1 \\ Buf_1 & \stackrel{def}{=} & (get, \top).Buf_0 + (put, \top).Buf_2 \\ Sys & \stackrel{def}{=} & Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1 \end{array}$$

$$\dfrac{\dfrac{Prod_1 \xrightarrow{(make, r_m)} Prod_2}{Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{get\}}{\bowtie} Prod_1 \xrightarrow{(make, r_m)} Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_2}}{Sys \xrightarrow{(make, r_m)} Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_2}$$

Summarising, the following transitions were found:

$$Sys \xrightarrow{(get, r_g)} Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1$$

$$Sys \xrightarrow{(make, r_m)} Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_2$$

Consumer/Producer in PEPA

$$\begin{array}{rclrcl} Cons_1 & \stackrel{def}{=} & (get, r_g).Cons_2 & Prod_1 & \stackrel{def}{=} & (make, r_m).Prod_2 \\ Cons_2 & \stackrel{def}{=} & (cons, r_c).Cons_1 & Prod_2 & \stackrel{def}{=} & (put, r_p).Prod_1 \\ Buf_2 & \stackrel{def}{=} & (get, \top).Buf_1 & Buf_1 & \stackrel{def}{=} & (get, \top).Buf_0 + (put, \top).Buf_2 \\ Buf_0 & \stackrel{def}{=} & (put, \top).Buf_1 & Sys & \stackrel{def}{=} & Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1 \end{array}$$

Popping $Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1$ off the stack,

$$\frac{Cons_2 \xrightarrow{(cons, r_c)} Cons_1}{Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1 \xrightarrow{(cons, r_c)} Cons_1 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1},$$

$$\frac{Prod_1 \xrightarrow{(make, r_m)} Prod_2}{Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1 \xrightarrow{(make, r_m)} Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_2}.$$

Consumer/Producer in PEPA

$$
\begin{array}{rclrcl}
Cons_1 & \stackrel{def}{=} & (get, r_g).Cons_2 & Prod_1 & \stackrel{def}{=} & (make, r_m).Prod_2 \\
Cons_2 & \stackrel{def}{=} & (cons, r_c).Cons_1 & Prod_2 & \stackrel{def}{=} & (put, r_p).Prod_1 \\
Buf_2 & \stackrel{def}{=} & (get, \top).Buf_1 & Buf_1 & \stackrel{def}{=} & (get, \top).Buf_0 + (put, \top).Buf_2 \\
Buf_0 & \stackrel{def}{=} & (put, \top).Buf_1 & Sys & \stackrel{def}{=} & Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_1
\end{array}
$$

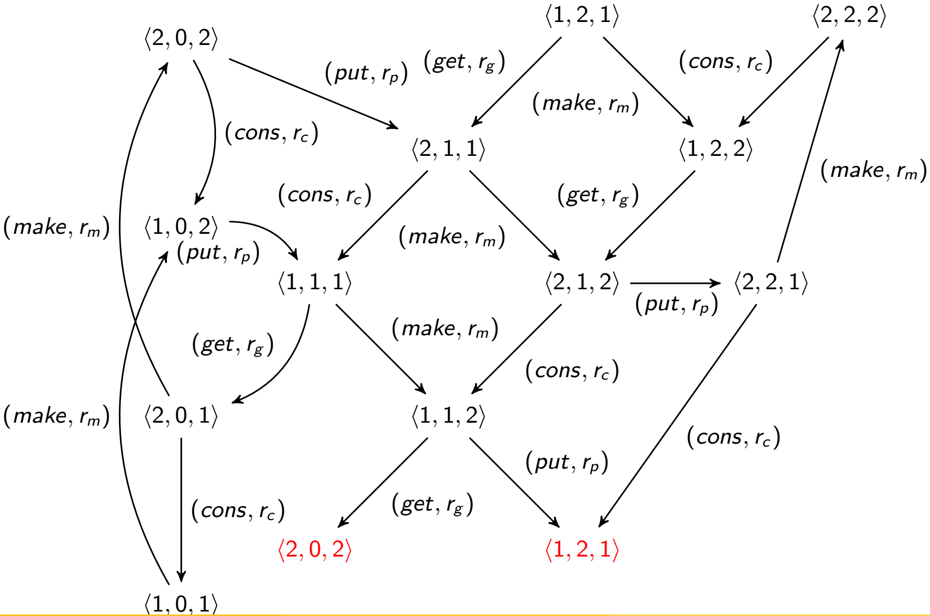
Therefore, we still need to infer transitions for the following processes...

$$
\begin{array}{c}
Cons_1 \underset{\{get\}}{\bowtie} Buf_2 \underset{\{put\}}{\bowtie} Prod_2 \\
Cons_1 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_1 \\
Cons_2 \underset{\{get\}}{\bowtie} Buf_1 \underset{\{put\}}{\bowtie} Prod_2
\end{array}
$$

...and all those that are found along the way.

Notice that the cooperation structure is fixed across all processes. Thus, we may denote a state by $\langle i, j, k \rangle$ to indicate $Cons_i \underset{\{get\}}{\bowtie} Buf_j \underset{\{put\}}{\bowtie} Prod_k$.

Consumer/Producer in PEPA: Complete Derivation Graph



Consumer/Producer in PEPA: State-Transition Diagram