

Performance Modelling of Computer Systems

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Lumpability

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Let X be a homogeneous, finite and irreducible CTMC with states $S = \{1, \dots, n\}$ and let the transition rate from i to j be denoted by $q(i, j)$. Let $\chi = \{X_1, \dots, X_N\}$ be a **partition** of the state space, with

$$X_i \subseteq S, \quad X_i \neq \emptyset, \text{ for all } i, \quad X_i \cap X_j = \emptyset, \text{ for all } i \neq j, \quad \text{and} \quad \bigcup_{i=1}^N X_i = S.$$

We say that X is **lumpable** with respect to χ if, for any $X_i, X_j \in \chi$ and $i_1, i_2 \in X_i$, it holds that

$$\sum_{j \in X_j} q(i_1, j) = \sum_{j \in X_j} q(i_2, j).$$

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Theorem

χ yields an aggregated (lumped) CTMC X' with states $\{X_1, \dots, X_N\}$ and

$$q(X_i, X_j) := \sum_{j \in X_j} q(i, j), \quad \text{for an arbitrary } i \in X_i.$$

Theorem

Let (π_1, \dots, π_n) denote the stationary distribution of X and (π'_1, \dots, π'_N) the stationary distribution of X' . Then, it holds that

$$\pi'_j = \sum_{i \in X_j} \pi_i, \quad 1 \leq j \leq N.$$

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For instance, let us consider the model

$$\begin{array}{ll} C_0 \stackrel{\text{def}}{=} (r, u).C_1 & C_1 \stackrel{\text{def}}{=} (t, v).C_0 \\ S_0 \stackrel{\text{def}}{=} (r, w).S_1 & S_1 \stackrel{\text{def}}{=} (i, x).S_0 \end{array} \quad \text{Sys} := C_0[M] \boxtimes_{\{r\}} S_0[M].$$

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- $|ds(\text{Sys})| = 2^{N+M}$ grows exponentially in N and M . Numerical analysis via explicit state enumeration is infeasible.
- Often, for many performance analyses, it is sufficient to know the distributions of the populations of the sequential components C_0 , C_1 , S_0 , and S_1 .

Lumpability — An Example

Let us denote a state in $ds(\text{Sys})$ by $(i_1, \dots, i_N, j_1, \dots, j_M) \in \{0, 1\}^{N+M}$.
Observing that the populations are given by the function

$$(i_1, \dots, i_N, j_1, \dots, j_M) \mapsto (N - \sum_{k=1}^N i_k, \sum_{k=1}^N i_k, M - \sum_{k=1}^M j_k, \sum_{k=1}^M j_k),$$

suggests to construct the partition from the **equivalence relation**
 $\sim \subseteq ds(\text{Sys}) \times ds(\text{Sys})$ defined by

$$(i_1, \dots, i_N, j_1, \dots, j_M) \sim (i'_1, \dots, i'_N, j'_1, \dots, j'_M) :\Leftrightarrow \\ \sum_{k=1}^N i_k = \sum_{k=1}^N i'_k \wedge \sum_{k=1}^M j_k = \sum_{k=1}^M j'_k.$$

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Indeed, one can prove that the CTMC of Sys is lumpable with respect to the partition $ds(\text{Sys})/\sim$.

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$$q(i_1, X_j) := \sum_{j \in X_j} q(i_1, j) \stackrel{(!)}{=} \sum_{j \in X_j} q(i_2, j) =: q(i_2, X_j).$$

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Together with $k \in \{1, 2\}$, $X_i = [(C_0^i, C_1^i, S_0^i, S_1^i)]$, $X_j = [(C_0^j, C_1^j, S_0^j, S_1^j)]$ we can infer

- case $[(C_0^j, C_1^j, S_0^j, S_1^j)] = [(C_0^i - 1, C_1^i + 1, S_0^i - 1, S_1^i + 1)]$:

$$q(i_k, X_j) = \left(\frac{u}{uC_0^i} \cdot \frac{w}{wS_0^i} \min(uC_0^i, wS_0^i) \right) \cdot (C_0^i \cdot S_0^i) = \min(uC_0^i, wS_0^i)$$

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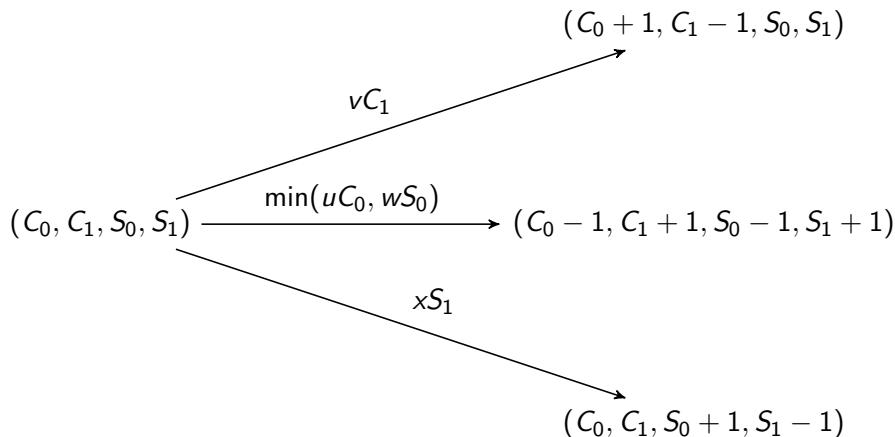
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- otherwise: $q(i_k, X_j) = 0$

Lumpability — An Example



- The transitions out of a state in the lumped CTMC.
- The lumped CTMC has $(N + 1)(M + 1)$ states.