Performance Modelling of Computer Systems
Tutorial 1

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Causal Independence vs Stochastic Independence

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The following model from Wikipedia illustrates this. Consider the probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\{1, \ldots, 6\}, \mathrm{Pot}(\Omega), \mathcal{U}_\Omega)$$

which models a fair die and the events

- $A :=$ the die gives an even number
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Then it is clear, that A and B are related to each other. But A and B are at the same time stochastically independent since

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(\text{the die gives an even number which is divisible by 3}) \\ &= \mathbb{P}(\{6\}) = \frac{1}{6} = \frac{1}{2}\frac{1}{3} = \mathbb{P}(A)\mathbb{P}(B) .\end{aligned}$$

Causal Independence vs Stochastic Independence

- Note that an event A is stochastically independent from itself iff

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A) \Longleftrightarrow \mathbb{P}(A) \in \{0, 1\} \ .$$

  This gives another (quite trivial) example where stochastic independence does not imply the causal independence.

- Another interesting question is the following one: let us assume that A_1, A_2 and A_3 are *pairwise* stochastic independent. Does this imply then their usual stochastic independence? (the answer is no).

A Formula which relies on Conditional Probabilities

Let us fix three events A_1, A_2, A_3 and assume, that we know the probabilities $\mathbb{P}(A_3|A_2 \cap A_1)$, $\mathbb{P}(A_2|A_1)$ and $\mathbb{P}(A_1)$. Multiplying yields then

$$\begin{aligned}\mathbb{P}(A_3|A_2 \cap A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_1) &= \frac{\mathbb{P}(A_3 \cap A_2 \cap A_1)}{\mathbb{P}(A_2 \cap A_1)} \frac{\mathbb{P}(A_2 \cap A_1)}{\mathbb{P}(A_1)} \mathbb{P}(A_1) \\ &= \mathbb{P}(A_3 \cap A_2 \cap A_1) \ .\end{aligned}$$

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This generalizes then to (can be proven by induction on n):

$$\begin{aligned}&\mathbb{P}(A_n \cap \ldots \cap A_1) = \\ &\qquad \mathbb{P}(A_n|A_{n-1} \cap \ldots \cap A_1)\mathbb{P}(A_{n-1}|A_{n-2} \cap \ldots \cap A_1) \cdot \ldots \cdot \mathbb{P}(A_1) \ .\end{aligned}$$

Exercise

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We define

$$A_i := \text{the } i\text{-th randomly chosen machine is fast}$$

and infer using the formula from last slide

$$\mathbb{P}(A_3 \cap A_2 \cap A_1) = \mathbb{P}(A_3 | A_2 \cap A_1)\mathbb{P}(A_2 | A_1)\mathbb{P}(A_1) = \frac{8}{48}\frac{9}{49}\frac{10}{50} \ .$$

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In fact, we can also observe that

$$\frac{8}{48}\frac{9}{49}\frac{10}{50} = \frac{10 \cdot 9 \cdot 8}{6}\frac{6}{50 \cdot 49 \cdot 48} = \frac{\binom{10}{3}}{\binom{50}{3}} .$$

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We infer then

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c) \\ &= \frac{19}{99}\frac{20}{100} + \frac{20}{99}\frac{80}{100} \\ &= \frac{1}{5}\end{aligned}$$

Exercise

- Under which conditions is the following a legitimate joint probability mass function?

	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 0$	a	$2a$	$2a$	a
$y = 1$	b	$2b$	$2b$	b

Exercise

- Let X be a continuous random variable with uniform distribution in $(0,1)$ and $Y = -\ln(1-X)/\lambda$, with $\lambda > 0$. Compute the cumulative distribution function of Y. (Random number generators)

Exercise

- Let $X_1, \ldots, X_n$ be independent exponentially distributed random variables. Prove then

$$\mathbb{P}(X_i = \min(X_1, \ldots, X_n)) = \frac{\lambda_i}{\lambda_1 + \ldots + \lambda_n} \ ,$$

for all $1 \leq i \leq n$ ($\lambda_j > 0$ denotes the parameter of X_j).

Exercise (the proof is not relevant for the exam)

The joint distribution of $X_1, \ldots, X_n$ is given by

$$\mathbb{P}(A) = \int_A \rho(\vec{x}) d\vec{x} = \int_A \Big(\prod_{i=1}^{n} \lambda_i e^{-\lambda_i x_i} \Big) d(x_1, \ldots, x_n) \ , \quad A \in \mathcal{B}(\mathbb{R}^n) \ .$$

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Since $\min(X_1, \ldots, X_n) = \min(X_1, \min(X_2, \ldots, X_n))$ and $\min(X_2, \ldots, X_n) \sim \mathrm{Exp}(\lambda_2 + \ldots + \lambda_n)$, it is sufficient to prove the claim for $n = 2$.

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Let us define $A := \{(x_1, x_2) \mid x_1 \leq x_2\}$. Then

$$\mathbb{P}(X_1 = \min(X_1, X_2)) = \mathbb{P}(X_1 \leq X_2) = \int_{\mathbb{R}^2} \rho(\vec{x}) \mathbb{1}_A(\vec{x}) d\vec{x}$$

$$\overset{\text{Fubini}}{=\!=} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \rho(x_1, x_2) \mathbb{1}_A(x_1, x_2) dx_2 \Big) dx_1$$

$$= \int_0^{\infty} \Big(\int_{x_1}^{\infty} \rho(x_1, x_2) dx_2 \Big) dx_1$$

$$\vdots$$

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$$
\begin{aligned}
&\vdots \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} \left(\int_{x_1}^\infty \lambda_2 e^{-\lambda_2 x_2} dx_2 \right) dx_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} \left[-e^{-\lambda_2 x_2} \right]_{x_1}^\infty dx_1 \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} \left(0 - (-e^{-\lambda_2 x_1}) \right) dx_1 \\
&= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2) x_1} dx_1 \\
&= \lambda_1 \left[-\frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2) x_1} \right]_0^\infty \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}
$$