Performance Modelling of Computer Systems
Tutorial 2

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Important DTMC Definitions

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- A state i of a DTMC has periodicity n if $p_{ii}^n > 0$.
- A state i of a DTMC is aperiodic if $gcd\{n \mid p_{ii}^n > 0\} = 1$.

Steady State Distributions and DTMCs

- A DTMC is irreducible if for every pair of states (i, j) there exists at least one path that leads from i to j. Note that this does not necessarily imply that all p_{ij} are non-zero.

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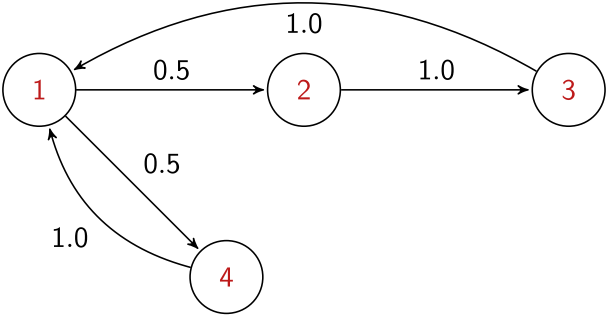
In the situation of performance modeling a DTMC is always finite. Also, in most cases it is irreducible (it could be reducible, if one wants to model deadlocks). Therefore, in order to show that there is a unique steady state distribution, we have usually to show that the DTMC is aperiodic.

Is the DTMC below ergodic?



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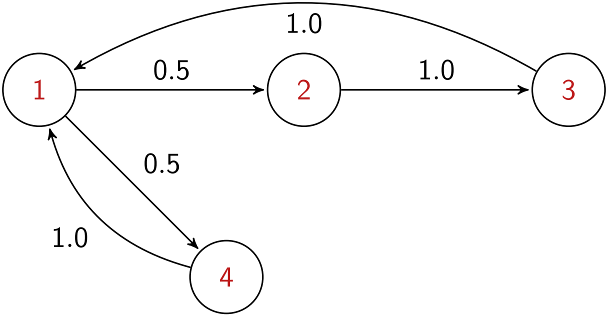

We note first that the DTMC is finite and irreducible. Observing then

- $1 \to 2 \to 3 \to 1$ and $1 \to 4 \to 1$ imply $2, 3 \in \mathcal{P}_1$
- $2 \to 3 \to 1 \to 2$ and $2 \to 3 \to 1 \to 4 \to 1 \to 2$ imply $3, 5 \in \mathcal{P}_2$
- $3 \to 1 \to 2 \to 3$ and $3 \to 1 \to 4 \to 1 \to 2 \to 3$ imply $3, 5 \in \mathcal{P}_3$
- $4 \to 1 \to 2 \to 3 \to 1 \to 4$ and $4 \to 1 \to 4$ imply $3, 5 \in \mathcal{P}_4$

for $\mathcal{P}_i := \{n \mid p_{ii}^n > 0\}$, implies then also the aperiodicity (and therefore the ergodicity) of the DTMC.

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$$P - I = \begin{pmatrix} -1 & 0.5 & 0 & 0.5 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} ,$$

the system we have to solve is

$$(\pi_1, \pi_2, \pi_3, \pi_4) \begin{pmatrix} -1 & 0.5 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (0, 0, 0, 1) .$$

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This yields then the (unique) solution $\pi = (0.4, 0.2, 0.2, 0.2)$.

Exercise

Given a DTMC $\{X_n, n \in \mathbb{N}\}$, prove that

$$\mathbb{P}(X_{n+2} = k, X_{n+1} = j \mid X_n = i) = p_{ij}(n)p_{jk}(n+1).$$

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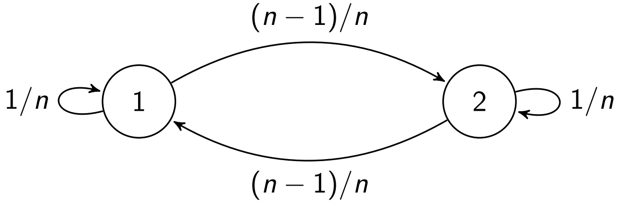
Using $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ and the strong Markov property we infer

$$\begin{aligned}
\mathbb{P}(X_{n+2} = k, &X_{n+1} = j \mid X_n = i) = \\
&= \frac{\mathbb{P}(X_{n+2} = k \mid X_{n+1} = j, X_n = i)\mathbb{P}(X_{n+1} = j, X_n = i)}{\mathbb{P}(X_n = i)} \\
&= \frac{\mathbb{P}(X_{n+2} = k \mid X_{n+1} = j)\mathbb{P}(X_{n+1} = j \mid X_n = i)\mathbb{P}(X_n = i)}{\mathbb{P}(X_n = i)} \\
&= \mathbb{P}(X_{n+2} = k \mid X_{n+1} = j)\mathbb{P}(X_{n+1} = j \mid X_n = i) \\
&= p_{ij}(n)p_{jk}(n+1)
\end{aligned}$$

Exercise

A server operates in two different conditions *Slow* (1) and *Fast* (2) according to the following *non-homogeneous* DTMC (the time step is given by n)



Find the probability that the server is slow at time step 4 given that it is fast at time step 1.

Exercise

Using the Chapman-Kolmogorov equations we derive

$$\begin{aligned}\mathbb{P}(X_4 = 1 | X_1 = 2) &= \sum_{(i_2,i_3)\in\{1,2\}^2} p_{(2,i_2)}(1) p_{(i_2,i_3)}(2) p_{(i_3,1)}(3) \\ &= p_{(2,1)}(1) p_{(1,1)}(2) p_{(1,1)}(3) + p_{(2,1)}(1) p_{(1,2)}(2) p_{(2,1)}(3) \\ &\quad + p_{(2,2)}(1) p_{(2,1)}(2) p_{(1,1)}(3) + p_{(2,2)}(1) p_{(2,2)}(2) p_{(2,1)}(3) \\ &= 0 + 0 + 1\frac{1}{2}\frac{1}{3} + 1\frac{1}{2}\frac{2}{3} \\ &= \frac{1}{2}\end{aligned}$$

Memorylessness and the Geometric Distribution

Recall that R is distributed w.r.t. the geometric distribution if

$$\mathbb{P}(R = n) = p^{n-1}(1 - p)$$

for all $n \geq 1$ and some fixed probability p.

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This is equivalent to

$$\begin{aligned}\mathbb{P}(R \leq n) &= \sum_{k=1}^{n} p^{k-1}(1-p) = (1-p)\sum_{k=0}^{n-1} p^k \\ &= (1-p)\frac{p^n - 1}{p-1} = -(p^n - 1) = 1 - (1 - \mathbb{P}(R = 1))^n \ .\end{aligned}$$

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Our goal it to prove that the memorylessness property

$$\mathbb{P}(R > m + n | R > m) = \mathbb{P}(R > n) \qquad \text{for all } m \geq 0, n > 0 \ ,$$

implies this equality, i.e., the geometric distribution is the only discrete distribution which satisfies the memorylessness property.

Memorylessness and the Geometric Distribution

Using the definition $G(l) := \mathbb{P}(R > l)$ we infer

$$\mathbb{P}(R > n) = \mathbb{P}(R > m + n | R > m) = \frac{\mathbb{P}(R > m + n \cap R > m)}{\mathbb{P}(R > m)} \quad \Leftrightarrow$$

$$\mathbb{P}(R > m + n) = \mathbb{P}(R > m)\mathbb{P}(R > n) \quad \Leftrightarrow$$

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and observe next

$$G(n) = G(n-1)G(1) = G(n-2)G(1)^2 = \ldots = G(1)^n \,.$$

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This yields then

$$\mathbb{P}(R \leq n) = 1 - \mathbb{P}(R > n) = 1 - \mathbb{P}(R > 1)^n = 1 - (1 - \mathbb{P}(R = 1))^n \, .$$

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