Performance Modelling of Computer Systems
Tutorial 4

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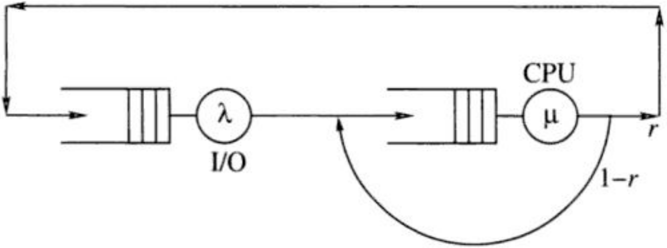
In a multiprogramming system, computer processes utilize a common CPU and an I/O device. After computing for an exponentially distributed amount of time with mean $1/\mu$, a process either joins the queue at the I/O device with probability r or exists the system with probability $1 - r$. At the I/O device, a process spends an exponentially distributed amount of time with mean $1/\lambda$ and then joins the CPU queue. The system is set up so that as soon as a process exists the system, a new process joins the CPU queue. Furthermore, we assume that the capacity of the system is equal to $K > 0$ and we start with K processes in the CPU queue.

Our system can be then expressed by the queueing network



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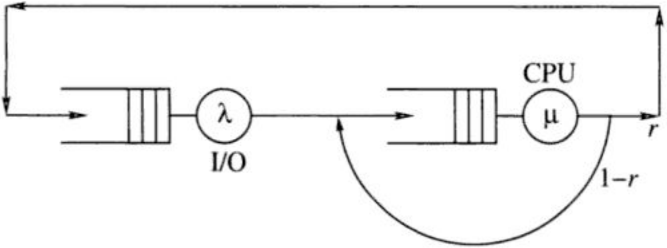
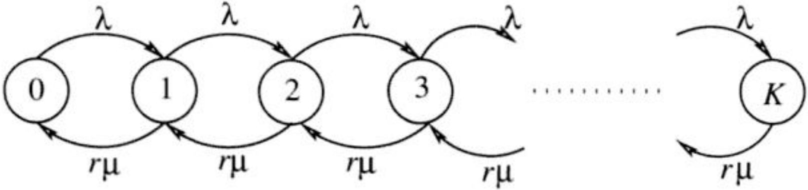


The sum of the waiting processes in both queues is equal to K at any time point. This fact allows us to translate this queueing network into the birth-death process



Setting $\rho := \frac{\lambda}{r\mu}$ and using the formulas from the lecture yields

$$\pi_k = \rho^k \pi_0 \ , \qquad k \geq 0$$

and

$$\pi_0 = \left(\sum_{i=0}^{K} \rho^k \right)^{-1} = \begin{cases} (1-\rho)\left(1-\rho^{K+1}\right) & , \ \rho \neq 1 \\ \dfrac{1}{K+1} & , \ \rho = 1 \end{cases}$$

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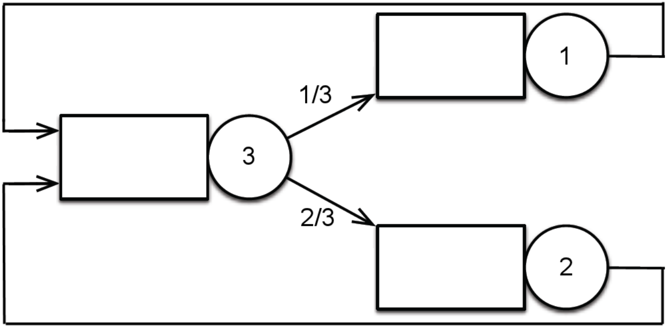
The CPU utilization is then

$$1 - \pi_0 = \begin{cases} \dfrac{\rho - \rho^{K+1}}{1-\rho^{K+1}} & , \ \rho \neq 1 \\ \dfrac{K}{K+1} & , \ \rho = 1 \end{cases}$$

Exercise

Calculate the steady state distribution and the performance measures of the closed queueing network below. We assume a single server policy with rates $\mu_1 := 3$, $\mu_2 := 1$, $\mu_3 := 2$ and a population of $N = 3$.



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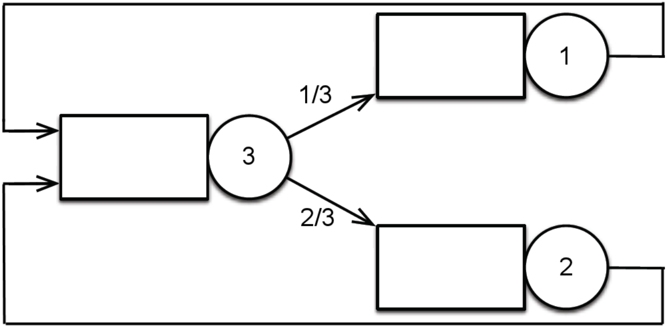

The size of the state space $\mathcal{S}(3,3) = \{(n_1, n_2, n_3) \mid \sum_{i=1}^{3} n_i = 3\}$ is $\binom{3+3-1}{3-1} = 10$. Solving $\vec{\lambda} = \vec{\lambda} Q$ yields

$$\begin{aligned} \lambda_1 &= \lambda_2 + \lambda_3 \\ \lambda_2 &= 1/3\lambda_1 \\ \lambda_3 &= 2/3\lambda_1 \end{aligned} \qquad \begin{gathered} \lambda_1 := 1 \\ \Longrightarrow \end{gathered} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1/3 \\ \lambda_3 &= 2/3 \end{aligned}$$

$$\lambda_1 = \lambda_2 + \lambda_3 \qquad \lambda_1 := 1 \qquad \lambda_1 = 1$$
$$\lambda_2 = 1/3\lambda_1 \qquad \Longrightarrow \qquad \lambda_2 = 1/3$$
$$\lambda_3 = 2/3\lambda_1 \qquad \qquad \lambda_3 = 2/3$$

Therefore

$$\begin{aligned}
G(3) = g_3(3) &= \sum_{k=0}^{3} f_3(k) g_2(3-k) \\
&= \sum_{k=0}^{3} f_3(k) \left(\sum_{k'=0}^{3-k} f_2(k') g_1(3-k-k') \right) \\
&= \sum_{k=0}^{3} \sum_{k'=0}^{3-k} f_3(k) f_2(k') f_1(3-(k+k')) \\
&= \sum_{k=0}^{3} \sum_{k'=0}^{3-k} \left(\frac{\lambda_1}{\mu_1}\right)^{k} \left(\frac{\lambda_2}{\mu_2}\right)^{k'} \left(\frac{\lambda_3}{\mu_3}\right)^{3-(k+k')} = \sum_{k=0}^{3} \sum_{k'=0}^{3-k} \frac{1}{27} = \frac{10}{27}
\end{aligned}$$

From this we infer then

$$\pi(n_1, n_2, n_3) = \frac{1}{G(3)} \prod_{i=1}^{3} f_i(n_i) = \frac{27}{10} \frac{1}{3^{(n_1+n_2+n_3)}} = \frac{27}{10 \cdot 27} = \frac{1}{10} \ ,$$

since $(n_1, n_2, n_3) \in \mathcal{S}(3,3)$. That is, $\pi = \mathcal{U}_{\mathcal{S}(3,3)}$.

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The performance measures $X_i(3)$ and $U_i(3)$ can be inferred from $G(3)$ and

$$\begin{aligned} G(2) = g_3(2) &= \sum_{k=0}^{2} f_3(k) g_2(2-k) \\ &= \sum_{k=0}^{2} f_3(k) \left(\sum_{k'=0}^{2-k} f_2(k') g_1(2-k-k') \right) \\ &= \sum_{k=0}^{2} \sum_{k'=0}^{2-k} f_3(k) f_2(k') f_1(2-(k+k')) \\ &= \sum_{k=0}^{2} \sum_{k'=0}^{2-k} \left(\frac{\lambda_1}{\mu_1}\right)^{k} \left(\frac{\lambda_2}{\mu_2}\right)^{k'} \left(\frac{\lambda_3}{\mu_3}\right)^{2-(k+k')} = \sum_{k=0}^{2} \sum_{k'=0}^{2-k} \frac{1}{9} = \frac{6}{9} = \frac{2}{3} \end{aligned}$$