Formal Techniques for Software Engineering: Behavioural Equivalences

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Lesson 8

Behavioural Equivalence

Specifications vs Implementations

- We are given an abstract system specification *Spec*
- We devise an implementation *Imp* by assembling many interacting components

A Natural Question

Are the processes *Imp* and *Spec* "behaviourally equivalent"?

the answer requires

- Fixing a "good" notion of equivalence
- Proving that the two processes are equivalent or finding a counterexample and re-design *Imp*...
- ... or changing the notion of equivalence

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Which Equivalence 1

Which processes should a reasonable behavioural equivalence equate?

- Two syntactic objects are equivalent if they have the same "meaning"
- Two processes are equivalent if they have the same "behavior", i.e., communication potential, as described by LTS's.

Idea:

Say the meaning of a process P is LTS(P), the LTS associated to it

But this yields too many distinctions:

$$X = a.X \qquad\qquad Y = a.a.Y$$

have different LTS but both processes can (only) execute infinitely many a-actions, and should be considered equivalent.

Which Equivalence 2

What should a reasonable behavioural equivalence satisfy?

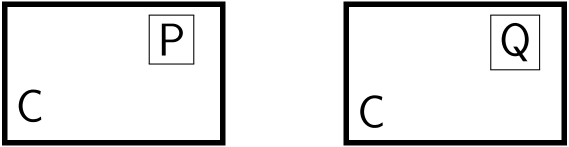
- Abstract from states (consider only the actions);
- abstract from internal behaviour (τ steps are not visible);
- identifies processes whose LTSs are isomorphic;
- considers two processes equivalent only if both can execute the same actions sequences;
- allows to replace a subprocess by an equivalent counterpart without changing the overall semantics of the system;
- be deadlock sensitive, i.e., if one has a deadlock after a given trace s, then then the other process has a deadlock after the same trace (and vice versa).

Which Equivalence 3

What else should a reasonable behavioural equivalence satisfy?

- Reflexivity: $P \equiv P$ for each process P
- Transitivity: $Spec_0 \equiv Spec_1 \equiv Spec_2 \equiv \cdots \equiv Impl$ gives that
  $$Spec_0 \equiv Impl$$
- Symmetry: $P \equiv Q$ iff $Q \equiv P$

An important property: Congruence



$C(P)$ $\qquad$ $C(Q)$

Congruence Property

$$P \equiv Q \text{ implies that } C(P) \equiv C(Q)$$

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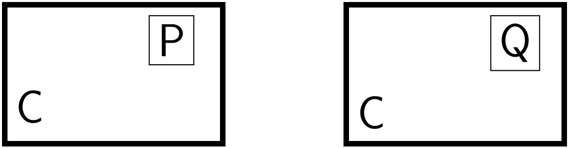

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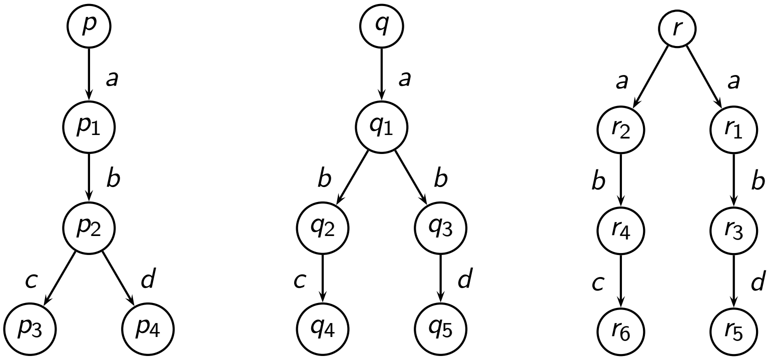

Problem: Should we consider these three systems as equivalent?

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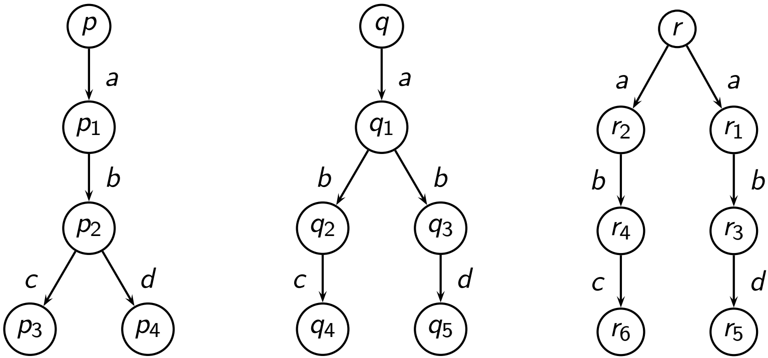

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Traces/Language Equivalence

Let $\langle Q, A, \rightarrow \rangle$ be an LTS, with $q \in Q$ and $s \in A^*$.

Traces

1. s is a *trace* of q if there exists $q' \in Q$ s.t. $q \xrightarrow{s} q'$.
2. $T(q)$ represents the set of all traces of q

Traces Equivalence

Two states p and q are *trace equivalent*, written $p =_T q$, if $T(p) = T(q)$.

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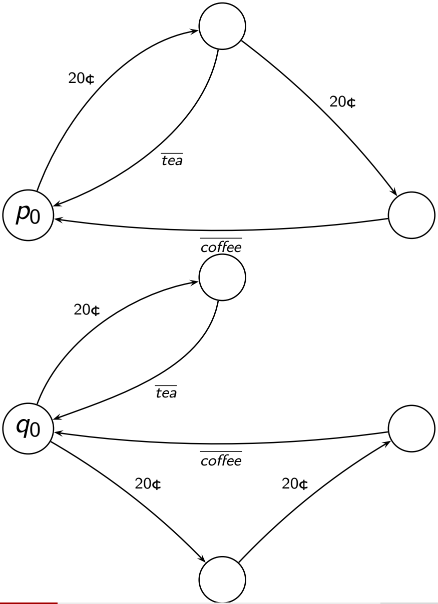

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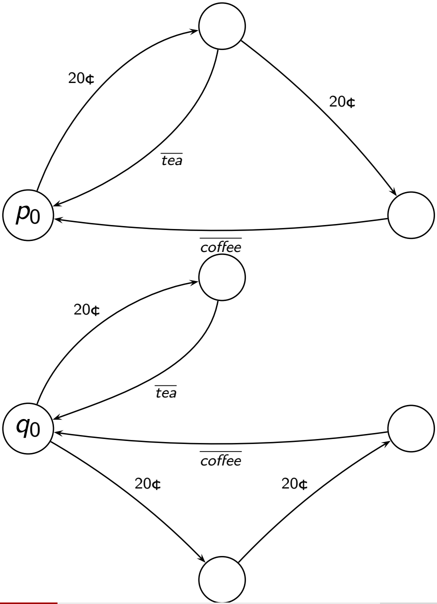

Black-Box Experiments

Experiment in A	Experiment in B	Experiment in B
coin $\overline{tea}$ $\overline{coffee}$	*coin* $\overline{tea}$ $\overline{coffee}$	*coin* $\overline{tea}$ $\overline{coffee}$
press *coin*	press *coin*	press *coin*
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Main Idea

Two processes are behaviorally equivalent if and only if an external observer cannot tell them apart.

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Bisimulation Relation

Strong Bisimulation

A relation $R \subseteq Q \times Q$ is *strong bisimulation* if, for any pair of states p and q such that $\langle p, q \rangle \in R$, the following holds:

1. for all $a \in A$ and $p' \in Q$, if $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ for some $q' \in Q$ such that $\langle p', q' \rangle \in R$;
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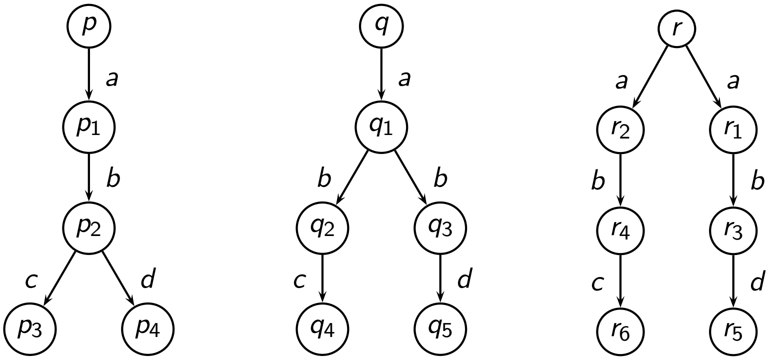

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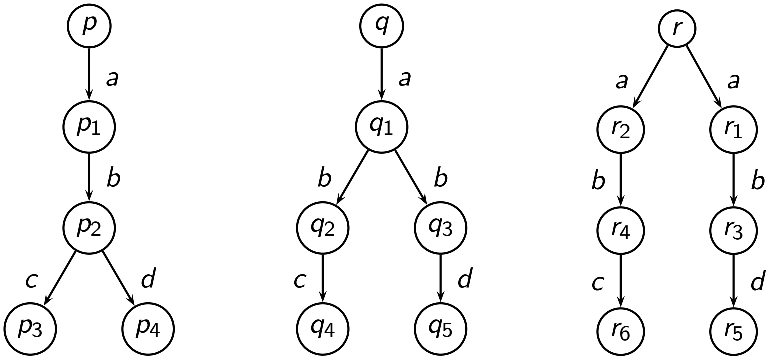

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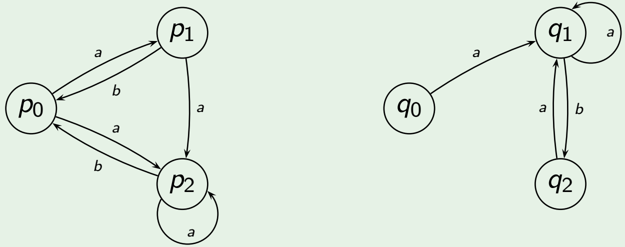
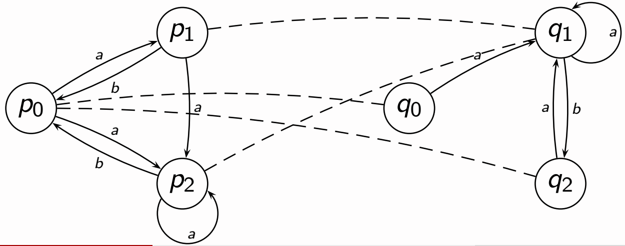


$R \triangleq \{\langle p_0, q_0\rangle, \langle p_0, q_2\rangle, \langle p_1, q_1\rangle, \langle p_2, q_1\rangle\}$ is a strong bisimulation



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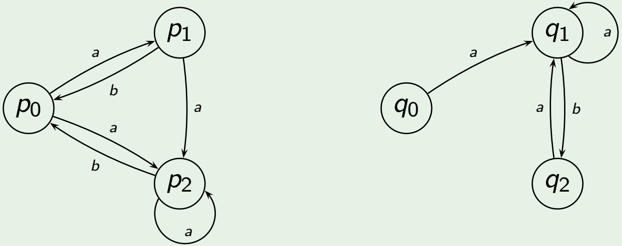

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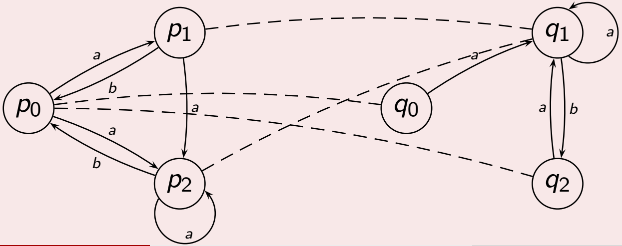

Basic Properties of Strong Bisimilarity

Theorem

$\sim$ is an equivalence relation (reflexive, symmetric and transitive)

Theorem

$\sim$ is the largest strong bisimulation

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$s \sim t$ if and only if for each $a \in Act$:

- *if $s \xrightarrow{a} s'$ then $t \xrightarrow{a} t'$ for some t' such that $s' \sim t'$*
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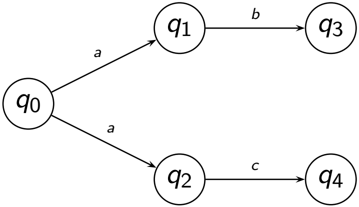

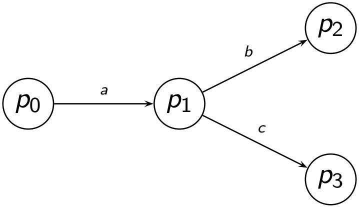

- States p_0 and q_0 are not strongly bisimilar.
- If they were equivalent, also states p_1 and q_1, had to be so.
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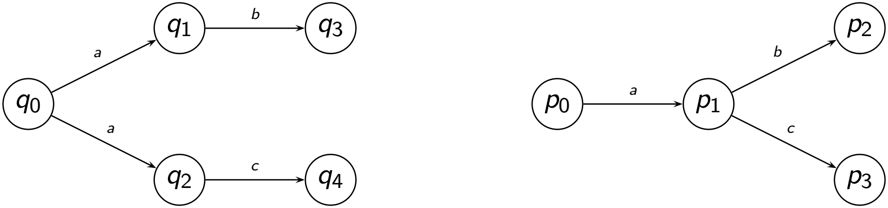

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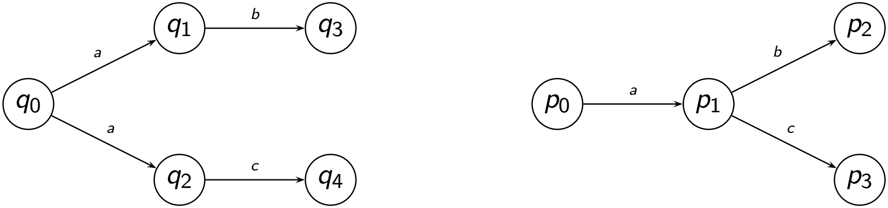

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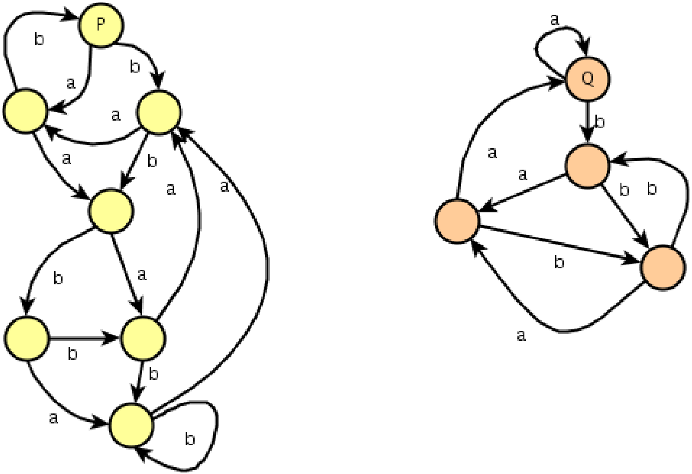
Are P and Q bisimilar?



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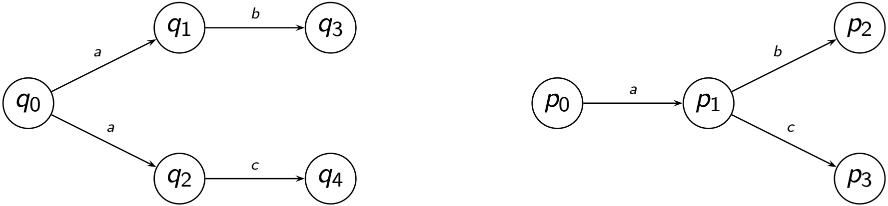

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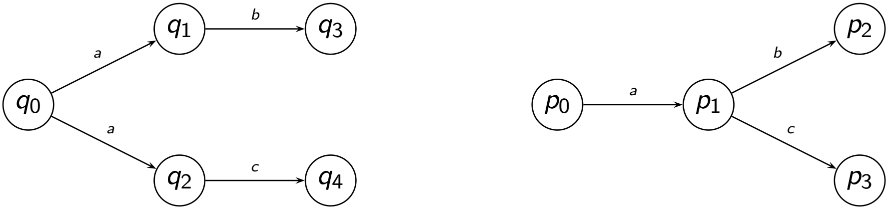

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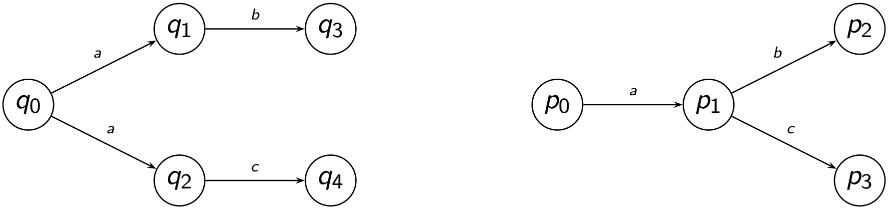

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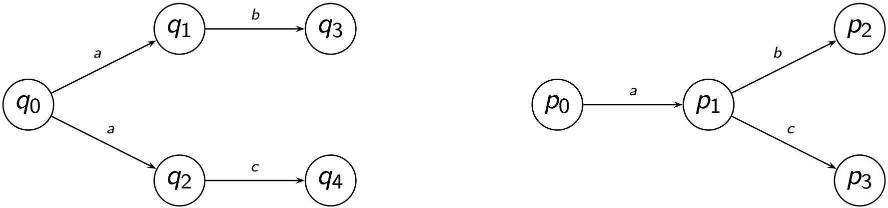

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Strong Bisimulation Game

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS and $s, t \in Proc$.

We define a two-player game of an 'attacker' and a 'defender' starting from s and t.

- The game is played in rounds, and configurations of the game are pairs of states from $Proc \times Proc$.
- In every round exactly one configuration is called current. Initially the configuration (s, t) is the current one.

Intuition

The defender wants to show that s and t are strongly bisimilar while the attacker aims at proving the opposite.

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Rules of the Bisimulation Games

Game Rules

In each round the players change the current configuration as follows:

1. the attacker chooses one of the processes in the current configuration and makes an $\xrightarrow{a}$-move for some $a \in Act$, and
2. the defender must respond by making an $\xrightarrow{a}$-move in the other process under the same action a.

The newly reached pair of processes becomes the current configuration. The game then continues by another round.

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- If one player cannot move, the other player wins.
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Theorem

- States s and t are strongly bisimilar if and only if the defender has a universal winning strategy starting from the configuration (s, t).
- States s and t are not strongly bisimilar if and only if the attacker has a universal winning strategy starting from the configuration (s, t).

Remark

The bisimulation game can be used to prove both bisimilarity and nonbisimilarity of two processes. It very often provides elegant arguments for the negative case.

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Simulation Relation

Strong Simulation

A relation $R \subseteq Q \times Q$ is *strong simulation* if, for any pair of states p and q such that $\langle p, q \rangle \in R$, the following holds:

- for all $a \in A$ and $p' \in Q$, if $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ for some $q' \in Q$ such that $\langle p', q' \rangle \in R$;

Similarity

Two states $p, q \in Q$ are strongly *similar*, written $p \sqsubseteq q$, if there exists a strong simulation R such that $\langle p, q \rangle \in R$.

$$\sqsubseteq \; = \bigcup \{ R \mid R \text{ is a strong simulation} \}$$

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More on simulation relation

- Is $\sqsubseteq$ a preorder?
- Is $\sqsubseteq$ an equivalence?
- Let S be a strong bisimulation. Is S a strong simulation? And S^{-1}?
- Let S be a strong simulation s.t. S^{-1} is also a strong simulation. Is S a strong bisimulation?
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A General Observational Approach

When defining behavioural equivalences, we do not want to distinguish between systems that cannot be taken apart by external observers.
Because of this we introduce three notions

1. Observers
2. Observations
3. Successful Observations

Those systems that satisfy (lead to successful observations) the same observers are considered equivalent.

- An *observer* is an LTS having actions from $A_w \triangleq A \cup \{w\}$, with $w \notin A$;
- To determine whether a state q satisfies an observer o the set $OBS(q, o)$ of all *computations* from $\langle q, o \rangle$ is considered
- A process may satisfy an observer always or sometimes.

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May Equivalence

p is *may* equivalent to q, $p \simeq_m q$, if for all observers $o \in \mathcal{O}$ we have p MAY SATISFY o if and only if q MAY SATISFY o;

Must Equivalence

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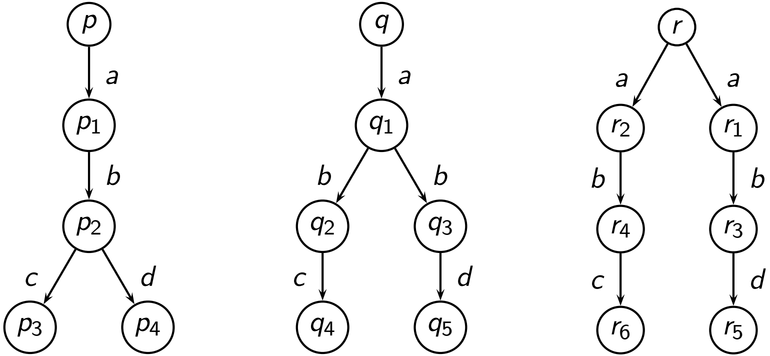

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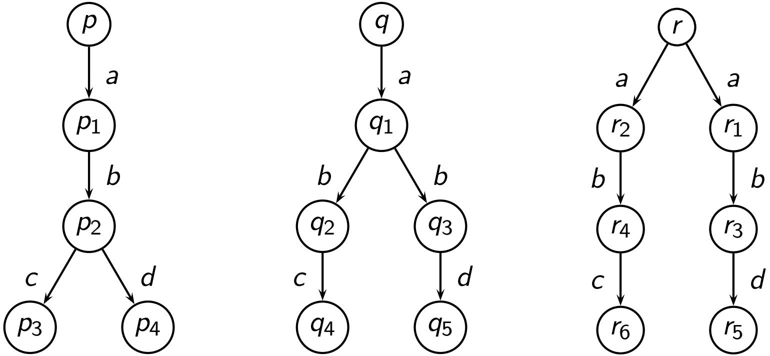

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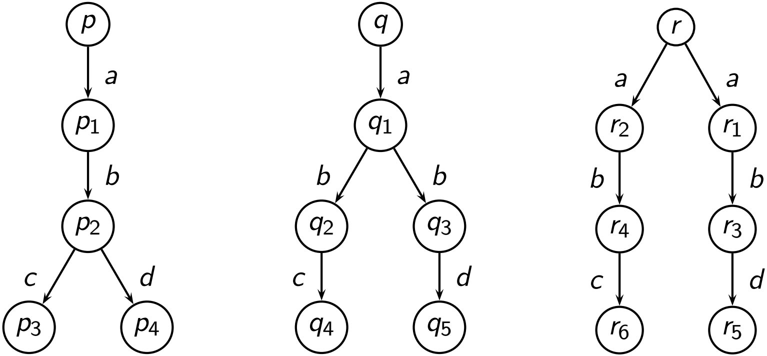

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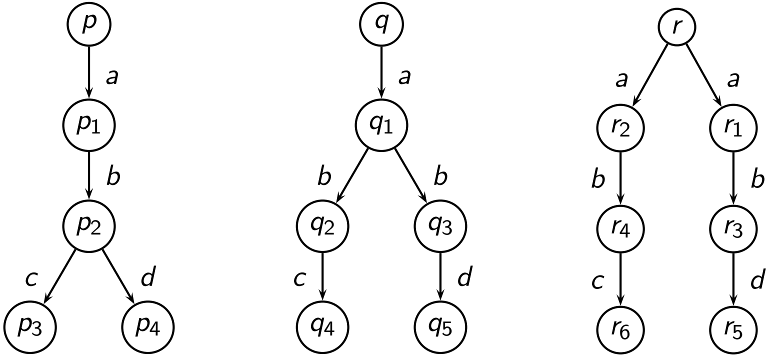

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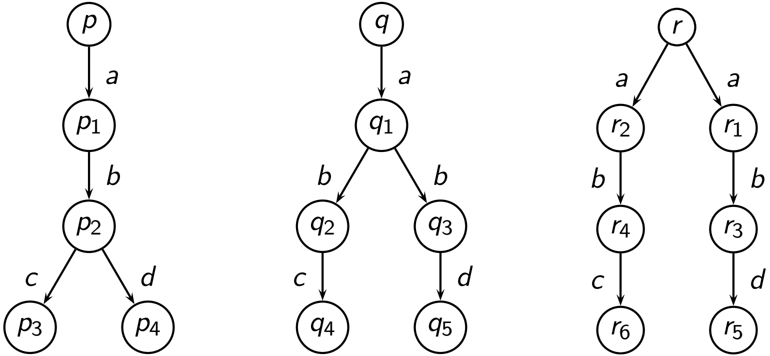

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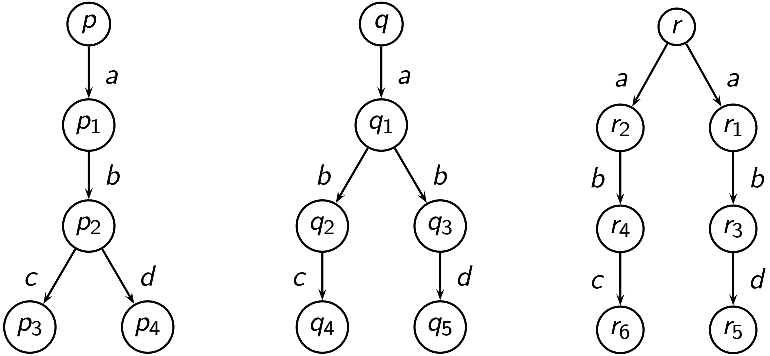

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Alternative characterisations of Testing Equivalences

Failures

Let $\langle Q, A_\tau, \rightarrow \rangle$ be an LTS, $q \in Q$, $s \in A^*$ and $L \subseteq A$

1. q *refuses* L if for all $b \in B$, there is no q' such that $q \xrightarrow{b} q'$.
2. $\langle s, L \rangle$ is a *failure* for q if there exists q' such that $q \xrightarrow{s} q'$ and q' refuses L.

Failures Equivalence

Let $\mathcal{F}(q)$ denote the set of all failures of a generic state q then

- $p \simeq_{\mathcal{F}} r$ if and only if $\mathcal{F}(p) = \mathcal{F}(r)$

Alternative May and Must Equivalences

- $p \simeq_M q$ if and only if $p \simeq_{\mathcal{F}} q$
- $p \simeq_m q$ if and only if $p \simeq_T q$

Another characterisation of Must Testing

Futures

Let $\langle Q, A_\tau, \rightarrow \rangle$ be an LTS, $q \in Q$ and $s \in A^*$

- $q \ after \ s = \{q' \mid q \xrightarrow{s} q'\}$
- $\text{Init}(q) = \{a \mid q \xrightarrow{a} q'\}$

Acceptance Sets

Let $\langle Q, A_\tau, \rightarrow \rangle$ be an LTS, $p \in Q$, $s \in A^*$ and $L \subseteq A$, $P \subseteq Q$

- p MUST L if and only if $\text{Init}(p) \cap L \neq \emptyset$
- P MUST L if and only if $\forall p \in P \ p$ MUST L

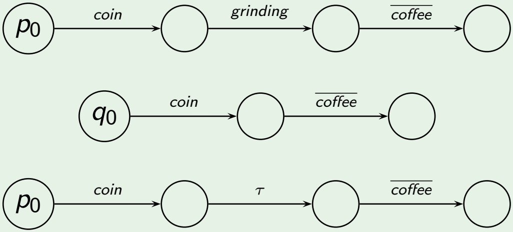
Alternative Must Equivalence

- $p \simeq_M q$ if and only if $\forall L, \forall s$
  p after s MUST L iff q after s MUST L

Weak Equivalences

Is it right to consider different from a user point of view the three machine below, if

- grinding is an internal action?
- τ is an invisible action?



Weak Traces Equivalence

Let $\langle Q, A, \rightarrow \rangle$ be an LTS, with $q \in Q$ and $s \in A^*$
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Let $q \stackrel{s}{\Rightarrow} q'$ denote that q reduces to q' by performing the sequence s of visible actions each of which can be preceded or followed by internal actions τ.

Weak Traces

1. s is a *weak trace* of q if there exists $q' \in Q$ s.t. $q \stackrel{s}{\Rightarrow} q'$.
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Weak Observations

To define the weak variants of may, must and testing equivalences it suffices to change experiments so that processes and observers can freely perform silent actions

Given two LTS $\langle Q, A_\tau, \rightarrow \rangle$ and $\langle O, A_{\tau,w}, \rightarrow \rangle$, and two states $q \in Q$ and $o \in O$, a weak *experiment* c from $\langle q, o \rangle$ is a sequence of pairs $\langle q_i, o_i \rangle$,s.t.

1. $\langle q_0, o_0 \rangle = \langle q, o \rangle$;
2. the transition $\langle q_i, o_i \rangle \xrightarrow{\mu} \langle q_{i+1}, o_{i+1} \rangle$ can be proved using:

$$\frac{E \xrightarrow{\tau} E'}{\langle E, F \rangle \xrightarrow{\tau} \langle E', F \rangle} \qquad \frac{F \xrightarrow{\tau} F'}{\langle E, F \rangle \xrightarrow{\tau} \langle E, F' \rangle} \qquad \frac{E \xrightarrow{a} E' \quad F \xrightarrow{a} F'}{\langle E, F \rangle \xrightarrow{\tau} \langle E', F' \rangle}\; a \in A$$

3. the last element of the sequence, say $\langle q_k, o_k \rangle$, is such that for no configuration $\langle q', o' \rangle$, with $q' \in Q$ and $o' \in O$, there exists a transition $\langle q_k, o_k \rangle \xrightarrow{\tau} \langle q', o' \rangle$ via the above rule.

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Weak Testing

Futures

Let $\langle Q, A_\tau, \rightarrow \rangle$ be an LTS, $q \in Q$ and $s \in A^*$

- q *after* $s = \{q' \mid q \stackrel{s}{\Rightarrow} q'\}$
- $\text{Init}(q) = \{a \mid q \stackrel{a}{\Rightarrow} q'\}$

Acceptance Sets

Let $\langle Q, A_\tau, \rightarrow \rangle$ be an LTS, $p \in Q$, $s \in A^*$ and $L \subseteq A$, $P \subseteq Q$

- p MUST L if and only if $\text{Init}(p) \cap L \neq \emptyset$
- P MUST L if and only if $\forall p \in P$ p MUST L

Alternative Must Equivalence

- $p \simeq_M q$ if and only if $\forall L, \forall s$
  p after s MUST L iff q after s MUST L

Weak Bisimulation Relation: An immediate generalization

Weak Bisimulation

A relation $R \subseteq Q \times Q$ is *weak bisimulation* if, for any pair of states p and q such that $\langle p, q \rangle \in R$, the following holds:

1. for all $s \in A^*$ and $p' \in Q$, if $p \stackrel{s}{\Rightarrow} p'$ then $q \stackrel{s}{\Rightarrow} q'$ for some $q' \in Q$ such that $\langle p', q' \rangle \in R$;
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Weak Bisimilarity

Two states $p, q \in Q$ of an LTS $\langle Q, A_\tau, \rightarrow \rangle$ are *weakly bisimilar*, written $p \approx q$, if there exists a weak bisimulation R such that $\langle p, q \rangle \in R$.

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where

$$\hat{\mu} = \begin{cases} \epsilon & \text{se } \mu = \tau \\ \mu & \text{se } \mu \neq \tau \end{cases}$$

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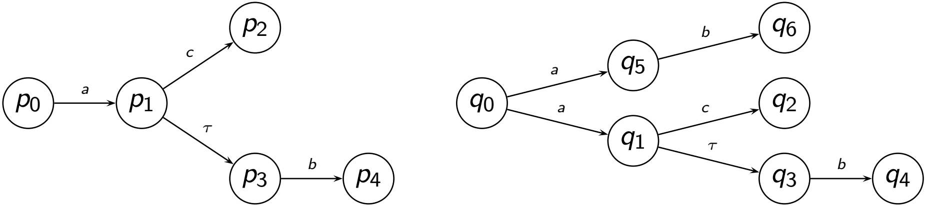
Two Pairs of Weakly Bisimilar Systems

Ignoring Tau's

Ignoring Tau's and Branching



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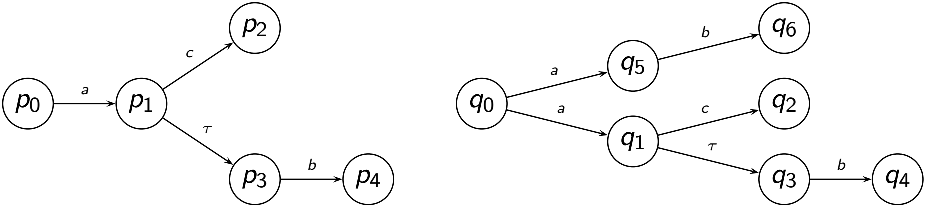

An Alternative to Weak Bisimulation

Branching Bisimulation

A symmetric relation $R \subseteq Q \times Q$ is *branching bisimulation* if, for any pair of states p and q such that $\langle p, q \rangle \in R$, if $p \xrightarrow{\mu} p'$, with $\mu \in A_\tau$ and $p' \in Q$, at least one of the following conditions holds:

1. $\mu = \tau$ and $\langle p', q \rangle \in R$
2. $q \Rightarrow q'' \xrightarrow{\mu} q'$ for some $q', q'' \in Q$ such that $\langle p, q'' \rangle \in R$ and $\langle p', q' \rangle \in R$.

Branching Bisimilarity

Two states $p, q \in Q$ of an LTS $\langle Q, A_\tau, \longrightarrow \rangle$ are *Branching bisimilar*, written $p \approx_b q$, if there exists a branching bisimulation R such that $\langle p, q \rangle \in R$.

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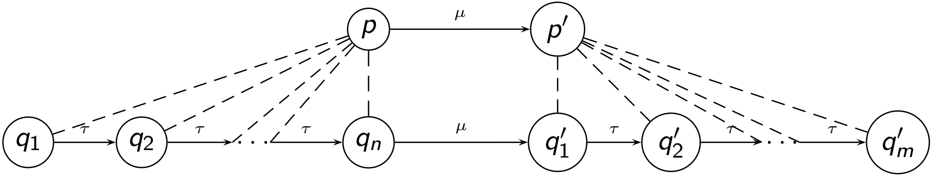
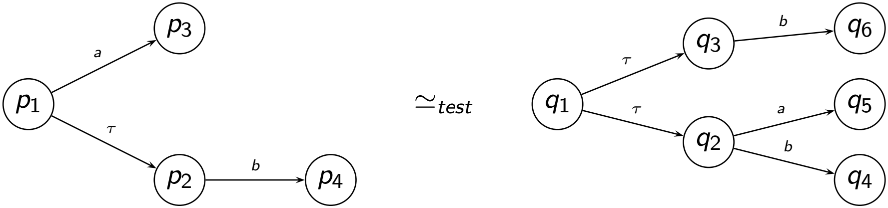
Branching Bisimulation, ... pictorially

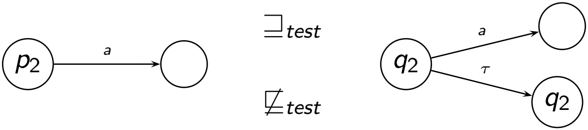


Testing vs Bisimulation - 1



The systems above are weakly testing equivalent but NOT weakly (nor branching) bisimilar

Testing vs Bisimulation - 2



The systems above are NOT testing equivalent but are weakly (and branching) bisimilar.

N.B.: The τ-arrow from q_2 to q_2 denotes a τ-loop.

Equivalences Hierarchies

For strongly convergent systems (i.e., systems without tau-loops) we have: