An Example in SMALL (1/2)

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- Observe that $\mathcal{R}[\![\textbf{read}]\!]\rho_0\ \sigma_1 = \langle 5, \sigma_1[0/\mathit{lin}]\rangle \equiv \langle 5, \sigma_2\rangle$.
- With this, $\mathcal{D}[\![\textbf{var } x = \textbf{read}]\!]\rho_0\ \sigma_1 =$ $\big(\mathcal{R}[\![\textbf{read}]\!]\rho_0 \star \lambda v.\lambda\sigma.\langle\rho_0[\textbf{new } \sigma/x], \sigma[v/\textbf{new } \sigma]\rangle\big)\sigma_1 =$ $\langle\rho_0[\textbf{new } \sigma_2/x], \sigma_2[5/\textbf{new } \sigma_2]\rangle = \langle\rho_0[l/x], \sigma_2[5/l]\rangle \equiv \langle\rho_1, \sigma_3\rangle$

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- Hence, $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \mathcal{C}[\![x = x + 1]\!]\rho_1\ \sigma_3 =$ $\big(\mathcal{E}[\![x]\!]\rho_1 \star \textbf{ checkLOC } \star \lambda L.\mathcal{R}[\![x + 1]\!]\rho_1 \star \lambda v.\lambda\sigma.\sigma[v/L]\big)\sigma_3$

An Example in SMALL (2/2)

- Hence, $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \mathcal{C}[\![x = x+1]\!]\rho_1\ \sigma_3 =$
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An Example in SMALL (2/2)

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- $\mathcal{E}[\![x]\!]\rho_1\ \sigma_3 = \langle \rho_1(x), \sigma_3\rangle = \langle l, \sigma_3\rangle$ and **checkLOC** $l\ \sigma_3 = \langle l, \sigma_3\rangle$

An Example in SMALL (2/2)

- Hence, $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \mathcal{C}[\![x = x+1]\!]\rho_1\ \sigma_3 =$
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An Example in SMALL (2/2)

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- $\mathcal{R}[\![x]\!]\rho_1 = \mathcal{E}[\![x]\!]\rho_1 \star \lambda v.\lambda\sigma\ \textbf{cases}\ldots\textbf{endcases}$ yields $\mathcal{R}[\![x_1]\!]\rho_1\ \sigma_3 =$ $(\textbf{cases}\ldots\textbf{endcases})\rho_1(x)\ \sigma_3 = \langle \sigma_3(\rho_1(x)), \sigma_3\rangle = \langle 5, \sigma_3\rangle$

An Example in SMALL (2/2)

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- $\mathcal{R}[\![1]\!]\rho_1 = \mathcal{E}[\![1]\!]\rho_1 \star \lambda v.\lambda\sigma.$ **cases** ... **endcases** and $\mathcal{E}[\![1]\!]\rho_1 = \lambda\sigma.\langle 1, \sigma\rangle$ yield $\mathcal{R}[\![1]\!]\rho_1\ \sigma_3 = \langle 1, \sigma_3\rangle$

An Example in SMALL (2/2)

- Hence, $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \mathcal{C}[\![x = x+1]\!]\rho_1\ \sigma_3 =$ $\big(\mathcal{E}[\![x]\!]\rho_1 \star$ **checkLOC** $\star\, \lambda L.\mathcal{R}[\![x+1]\!]\rho_1 \star \lambda v.\lambda\sigma.\sigma[v/L]\big)\sigma_3$
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- Consequently, $\mathcal{R}[\![x+1]\!]\rho_1\ \sigma_3 = \langle 6, \sigma_3\rangle$ and we get $\lambda v.\lambda\sigma.\sigma[v/l]6\ \sigma_3 = \sigma_3[6/l]$

An Example in SMALL (2/2)

- Hence, $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \mathcal{C}[\![x = x+1]\!]\rho_1\ \sigma_3 = \big(\mathcal{E}[\![x]\!]\rho_1 \star\ \textbf{checkLOC}\ \star \lambda L.\mathcal{R}[\![x+1]\!]\rho_1 \star \lambda v.\lambda\sigma.\sigma[v/L]\big)\sigma_3$
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- $\mathcal{R}[\![x]\!]\rho_1 = \mathcal{E}[\![x]\!]\rho_1 \star \lambda v.\lambda\sigma\ \textbf{cases} \ldots \textbf{endcases}$ yields $\mathcal{R}[\![x_1]\!]\rho_1\ \sigma_3 = (\textbf{cases} \ldots \textbf{endcases})\rho_1(x)\ \sigma_3 = \langle \sigma_3(\rho_1(x)), \sigma_3\rangle = \langle 5, \sigma_3\rangle$
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- Consequently, $\mathcal{R}[\![x+1]\!]\rho_1\ \sigma_3 = \langle 6, \sigma_3\rangle$ and we get $\lambda v.\lambda\sigma.\sigma[v/l]6\ \sigma_3 = \sigma_3[6/l]$
- Therefore, it holds that $\mathcal{C}[\![c]\!]\rho_0\ \sigma_1 = \sigma_0[0/lin, nil/lout, 6/l]$ (and $\mathcal{P}[\![\textbf{program}\ c]\!]5 :: 0 = nil$).